A new construction of the simply laced simple Lie algebras from the affine Dynkin diagrams

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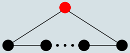
$\widetilde{A}_n$



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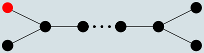

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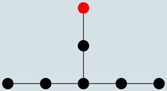
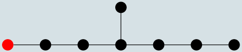
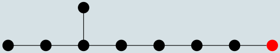


$\widetilde{E}_7$



$\widetilde{E}_8$



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Given: a finite simple connected graph $\Gamma = (\Pi, E)$ and a field $\mathbb{F}$.

Construct: $\mathcal{F}(\Gamma)$ the quotient of the free Lie algebra over $\mathbb{F}$ generated by Π modulo the relations

- $[x, y] = 0$ for all vertices $x \nsim y$.

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Graph $\longrightarrow$ Lie algebra with extremal generating elements

Given: The Lie algebra $\mathcal{F}(\Gamma)$ and $f := (f_x)_{x \in \Pi} \in (\mathcal{F}(\Gamma)^*)^\Pi$.

Construct: $\mathcal{L}(\Gamma, f)$ the quotient of $\mathcal{F}(\Gamma)$ by the ideal generated by the elements $[x, [x, y]] - f_x(y)x$ with $x \in \Pi$ and $y \in \mathcal{F}(\Gamma)$.

Upper bounds on the dimension

Theorem

- Zel'manov (1981), Zel'manov and Kostrikin (1991).

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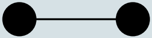
Corollary

$L(\Gamma, f)$ is a finite dimensional Lie algebra generated by extremal elements corresponding to the vertices of Γ.

An example

$\Gamma \quad =$ 

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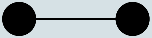
$\Gamma =$  $\Longrightarrow x, y, [x, y] \in \mathcal{L}(\Gamma, f).$

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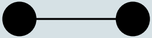
$\Gamma =$  $\Longrightarrow x, y, [x, y] \in \mathcal{L}(\Gamma, f).$

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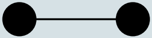
$\Gamma =$  $\Longrightarrow x, y, [x, y] \in \mathcal{L}(\Gamma, f).$

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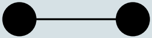
Γ =  $\Longrightarrow$ $x, y, [x, y] \in \mathcal{L}(\Gamma, f)$.

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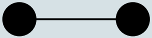
$\Gamma =$  $\Longrightarrow x, y, [x, y] \in \mathcal{L}(\Gamma, f).$

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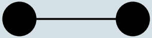
$\Gamma =$  $\Longrightarrow \quad x, y, [x, y] \in \mathcal{L}(\Gamma, f)$.

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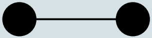
Γ =  $\Longrightarrow$ $x, y, [x, y] \in \mathcal{L}(\Gamma, f)$.

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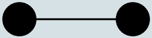
$\Gamma =$  $\Longrightarrow \quad x, y, [x,y] \in \mathcal{L}(\Gamma, f).$

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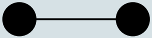
$\Gamma =$  $\implies x, y, [x, y] \in \mathcal{L}(\Gamma, f)$.

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The parameter space

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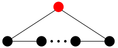

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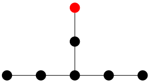

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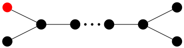

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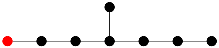

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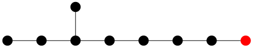

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Lemma

If $f \in X(\Gamma)$, then it is determined by the parameter values $f_x(y)$ ($\{x, y\} \in E$) and one additional value $f_z([z_d, [z_{d-1}, [\ldots, z_1]]])$ with $z, z_d, z_{d-1}, \ldots, z_1 \in \Pi$.

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Corollary

There is a closed embedding $\Psi : X(\Gamma) \hookrightarrow \mathbb{F}^E \times \mathbb{F}$ sending f to $\{f_x(y) \mid x \sim y\} \cup \{f_z([z_d, [z_{d-1}, [\ldots, z_1]]])\}$

Generic Lie algebras

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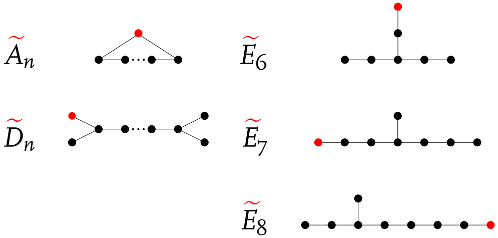

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Lemma

Let $\mathfrak{g}$ be the simply laced simple Lie algebra with finite Dynkin diagram $\mathring{\Gamma}$. Then there is an $f(\mathfrak{g}) \in X(\Gamma)$ such that $\mathcal{L}(\Gamma, f(\mathfrak{g})) \simeq \mathfrak{g}$ and $\Psi(f) \in (\mathbb{F}^*)^E \times \mathbb{F}^*$.

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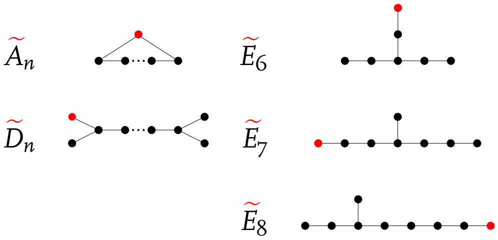

Scaling action

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$$(tf)_x(y) := t_x^{-1} f_x(t^{-1}y)),$$

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for all $t \in T,\ f \in X(\Gamma),\ x \in \Pi,$ and $y \in \mathcal{F}(\Gamma)$.
- The automorphism of $\mathcal{F}(\Gamma)$ induced by t induces an isomorphism $\mathcal{L}(\Gamma, f) \to \mathcal{L}(\Gamma, tf)$ for all $f \in X(\Gamma)$.

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- $X(\Gamma)$ is isomorphic to an affine space with linear action of T and the orbit of $f(\mathfrak{g})$ has codimension 0, 1 or 2.
- If $f \in X(\Gamma)$ and if $\Psi(f)$ is an element of some open dense subset of $\mathbb{F}^E \times \mathbb{F}$ then $\mathcal{L}(\Gamma, f)$ is isomorphic to $\mathfrak{g}$.

Questions?