On the p-adic Beilinson conjecture for number fields

Joint work with A. Besser, P. Buckingham and X.-F. Roblot

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The residue at $s = 1$ of $\zeta_k(s)$

k: a number field of degree d with ring of integers $\mathcal{O}_k$
r_1: the number of real embeddings of k
$2r_2$: the number of complex embeddings of k
$K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$ has rank $r = r_1 + r_2 - 1$
$\sigma_1, \ldots, \sigma_{r+1}$: the embeddings $k \to \mathbb{C}$ up to complex conjugation.
If $u_1, \ldots, u_r$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k^*$/torsion, let

$$R = \frac{2^{r_2}}{[k:\mathbb{Q}]} \left| \det \begin{pmatrix} 1 & \log|\sigma_1(u_1)| & \ldots & \log|\sigma_1(u_r)| \\ \vdots & \vdots & & \vdots \\ 1 & \log|\sigma_{r+1}(u_1)| & \ldots & \log|\sigma_{r+1}(u_r)| \end{pmatrix} \right|$$

Then $\mathrm{Res}_{s=1}\zeta_k(s) = 2^{r_1}(2\pi)^{r_2} R |\mathrm{Cl}(\mathcal{O}_k)| / (w\sqrt{|D_k|})$
[D_k = discriminant of k, w = #roots of unity in k]

Borel's theorem

Theorem (Quillen+Borel)

- for $n \geq 2$, $K_{2n-1}(k)$ is finitely generated;
- its rank m_n equals r_2 for n even, $r_1 + r_2$ for n odd;
- there is a natural regulator map

$$K_{2n-1}(k) \to \left(\prod_{\sigma : k \to \mathbb{C}} \mathbb{R}(n-1) \right)^+ \cong \mathbb{R}^{m_n} ,$$

where $\mathbb{R}(m) = (2\pi i)^m \mathbb{R} \subset \mathbb{C}$, and $+$ indicates those $(x_\sigma)_\sigma$ with $\overline{x_\sigma} = x_{\overline{\sigma}}$;

- the image is a lattice;
- if $V_n(k)$ is the volume of a fundamental domain of the image then $\zeta_k(n)\sqrt{|D_k|} = q_n \pi^{n(d-m_n)} V_n(k)$ for some q_n in $\mathbb{Q}^*$.

Example

If $k = \mathbb{Q}$ then $\zeta_{\mathbb{Q}}(s) = \zeta(s)$ and we get

n	2	3	4	5	$\cdots$
m_n	0	1	0	1	$\cdots$
$\zeta(n)$	$\pi^2/6$	$?$	$\pi^4/90$	$?$	$\cdots$

Interpolation formula

The p-adic L-function L_p of a totally real number field k satisfies

$$L_p(n, \omega_p^{1-n}, k) = E_p(n,k)\zeta_k(n)$$

for all integers $n \leq 0$, where $E_p(s,k) = \prod_{\mathcal{P}|p}(1 - N(\mathcal{P})^{-s})$. Here

$$\omega_p : Gal(\overline{k}/k) \longrightarrow \mu_{\phi(2p)} \subset \mathbb{Q}_p^*$$

is the Teichmüller character for k and p, i.e., the composition

$$Gal(\overline{k}/k) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Gal(\mathbb{Q}(\mu_{2p})/\mathbb{Q}) \cong (\mathbb{Z}/2p\mathbb{Z})^* \xrightarrow{\sim} \mu_{\phi(2p)}$$

A special case of the conjecture

Assume k is totally real, and take $n \geq 2$ odd.
Fix ordered $\mathbb{Q}$-bases $\{a_j\}$ of k and $\{\alpha_j\}$ of $K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$.
(Both have dimension d.)

Borel's map comes from composing 2 times the Beilinson regulator map $reg_\infty : K_{2n-1}(\mathbb{C}) \to \mathbb{R}(n-1)$ with $\sigma_*^\infty : K_{2n-1}(k) \to K_{2n-1}(\mathbb{C})$ for all embeddings $\sigma^\infty : k \to \mathbb{C}$.

Let $\sigma_1^\infty, \ldots, \sigma_d^\infty$ be the embeddings of k into $\mathbb{C}$.

With $D_k^{1/2,\infty} = \det(\sigma_i^\infty(a_j))$ and
$R_{n,\infty}(k) = \det(reg_\infty \circ \sigma_{i,*}^\infty(\alpha_j))$ Borel's theorem gives

$$\zeta_k(n) D_k^{1/2,\infty} = q(n,k) R_{n,\infty}(k)$$

for some $q(n,k)$ in $\mathbb{Q}^*$.

Let $F \subset \overline{\mathbb{Q}}_p$ be the topological closure of the Galois closure of k embedded into $\mathbb{Q}_p$ in any way.
There is a syntomic regulator $reg_p : K_{2n-1}(F) \to F$.
Let $\sigma_1^p, \dots, \sigma_d^p$ be the embeddings of k into F.

Put $D_k^{1/2,p} = \det(\sigma_i^p(a_j))$, $R_{n,p}(k) = \det(reg_p \circ \sigma_{i,*}^p(\alpha_j))$.

Conjecture (also special case of conjecture by Perrin-Riou)
(1) in F we have, for some $q_p(n,k)$ in $\mathbb{Q}^*$,

$$L_p(n, \omega_p^{1-n}, k) D_k^{1/2,p} = q_p(n,k) E_p(n,k) R_{n,p}(k) \, ;$$

(2) in fact, $q_p(n,k) = q(n,k)$;
(3) $L_p(n, \omega_p^{1-n}, k)$ and $R_{n,p}(k)$ are non-zero.

A simple(?) open problem

It is not known that $L_p(n, \omega_p^{1-n}, \mathbb{Q}) \neq 0$ when $n \geq 2$ is odd.

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For $p > 2$ this would certainly hold if

$$\sum_{a=1}^{p-1} (-1)^a (\overline{a})^{-n} = \overline{2}^{1-n} \sum_{b=1}^{\frac{p-1}{2}} (\overline{b})^{-n} \neq \overline{0} \text{ in } \mathbb{Z}/p\mathbb{Z}$$

This seems to be the case “often”. Is it the case for infinitely many p when n is fixed?

A motivic version

Assume $k/\mathbb{Q}$ is finite and Galois [but not necessarily totally real], with Galois group G.
Fix embeddings $\phi_\infty : k \to \mathbb{C}$ and $\phi_p : k \to F$.
$D_k^{1/2,?}$ and $R_{n,?}(k)$ for $? = \infty$ or p are determinants of pairings

$$(\,\cdot\,,\,\cdot\,)_? : \mathbb{Q}[G] \times k \longrightarrow \mathbb{C} \text{ or } F$$
$$(\sigma, a) \mapsto \phi_?(\sigma(a))$$

and

$$[\,\cdot\,,\,\cdot\,]_? : \mathbb{Q}[G] \times (K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}) \longrightarrow \mathbb{R}(n-1) \text{ or } F$$
$$(\sigma, \alpha) \mapsto reg_? \circ \phi_{?*}(\sigma_*(\alpha))$$

If $E/\mathbb{Q}$ is finite, π in $E[G]$ an idempotent, then we can tensor k and $K_{2n-1}(k)$ with E and consider E-bilinear pairings

$$(\cdot\,,\,\cdot)_? : E[G]\pi \times \pi(k \otimes_{\mathbb{Q}} E) \longrightarrow E \otimes_{\mathbb{Q}} \mathbb{C} \text{ or } E \otimes_{\mathbb{Q}} F$$

and

$$[\cdot\,,\,\cdot]_? : E[G]\pi \times \pi\big(K_{2n-1}(k) \otimes_{\mathbb{Z}} E\big) \longrightarrow E \otimes_{\mathbb{Q}} \mathbb{R}(n-1) \text{ or } E \otimes_{\mathbb{Q}} F$$

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The dimensions for $(\,\cdot\,,\,\cdot\,)_?$ always match, for $[\,\cdot\,,\,\cdot\,]_?$ they match in precisely two cases (for $n \geq 2$):
(1) the fixed field of the kernel of the representation of G on $E[G]\pi$ is totally real, and n is odd;
(2) the fixed field of the kernel of the representation of G on $E[G]\pi$ is CM, complex conjugation of that field acts on $E[G]\pi$ as multiplication by -1, and n is even.

Write M_π^E for $\pi(k \otimes_{\mathbb{Q}} E)$ and $K_{2n-1}(M_\pi^E)$ for $\pi(K_{2n-1}(k) \otimes_{\mathbb{Z}} E)$. Fix ordered E-bases of those spaces, as well as of $E[G]\pi$. Let $D(M_\pi^E)^{1/2,?}$ be the determinant of the resulting pairing $(\cdot\,,\,\cdot)_?$. In either case above, let $R_{n,?}(M_\pi^E)$ be the determinant of the resulting pairing $[\cdot\,,\,\cdot]_?$.

If χ_π is the character of the representation of G on $E[G]\pi$ one defines an $E \otimes_{\mathbb{Q}} \mathbb{C}$ valued L-function $L(s, \chi_\pi \otimes 1, \mathbb{Q})$, with an Euler product $L(s, \chi_\pi \otimes 1, \mathbb{Q}) = \prod_p E_p(s, \chi_\pi \otimes 1, \mathbb{Q})$.

In the two cases above there is also a meromorphic $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$-valued p-adic L-function, $L_p(s, \chi_\pi \otimes \omega_p^{1-n}, \mathbb{Q})$, together with an interpolation formula when $s \leq 0$ is an integer.

Conjecture (also . . .) In either case above, for $n \geq 2$:
(1) in $E \otimes_{\mathbb{Q}} \mathbb{C}$ we have

$$L(n, \chi_\pi \otimes 1, \mathbb{Q}) D(M_\pi^E)^{1/2,\infty} = e(n, M_\pi^E) R_{n,\infty}(M_\pi^E)$$

for some $e(n, M_\pi^E)$ in $(E \otimes_{\mathbb{Q}} \mathbb{Q})^*$;
(2) in $E \otimes_{\mathbb{Q}} F$ we have

$$L_p(n, \chi_\pi \otimes \omega_p^{1-n}, \mathbb{Q}) D(M_\pi^E)^{1/2,p} \\ = e_p(n, M_\pi^E)\, E_p(n, \chi_\pi \otimes 1, \mathbb{Q}) R_{n,p}(M_\pi^E)$$

for some $e_p(n, M_\pi^E)$ in $(E \otimes_{\mathbb{Q}} \mathbb{Q})^*$;
(3) in fact, $e_p(n, M_\pi^E) = e(n, M_\pi^E)$;
(4) $L_p(n, \chi_\pi \otimes \omega_p^{1-n}, \mathbb{Q})$ and $R_{n,p}(M_\pi^E)$ are units in $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $E \otimes_{\mathbb{Q}} F$ respectively.

Zagier's conjecture: describing $K_{2n-1}(k)$

Let $Li_n(z) = \sum_{j\geq 1} \frac{z^j}{j^n}$ $\quad (z$ in $\mathbb{C}$ with $|z| < 1; n \geq 0)$

- $Li_0(z) = z/(1-z)$
- $Li_1(z) = -\log(1-z)$
- $Li'_{n+1}(z) = Li_n(z)/z$
- $Li_n(z)$ extends to a multi-valued analytic function on $\mathbb{C} \setminus \{0,1\}$
- on $\mathbb{C} \setminus \{0,1\}$ $P_n(z) = \pi_{n-1}\left(\sum_{j=0}^{n-1} \frac{b_j}{j!}(2\log|z|)^j Li_{n-j}(z)\right)$ is single-valued and satisfies $P_n(z) + (-1)^n P_n(1/z) = 0$.

[$b_j = j$-th Bernoulli number; $\pi_m =$ projection onto $\mathbb{R}(m)$ in $\mathbb{C} = \mathbb{R}(m-1) \oplus \mathbb{R}(m)$.]

For $n \geq 2$:

- let $B_n(k)$ be a free abelian group on $[x]_n$ ($x \neq 0, 1$ in k)
- define

$$\tilde{P}_n : B_n(k) \longrightarrow \mathbb{R}(n-1)^d$$
$$[x]_n \mapsto (P_n(\sigma(x)))_{\sigma : k \to \mathbb{C}}$$

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- define inductively:

$$d_n : B_n(k) \to \begin{cases} \bigwedge^2_{\mathbb{Z}} k^* & \text{if } n = 2 \\ C_{n-1}(k) \otimes_{\mathbb{Z}} k^* & \text{if } n > 2 \end{cases}$$

$$[x]_n \mapsto \begin{cases} (1-x) \wedge x & \text{if } n = 2 \\ [x]_{n-1} \otimes x & \text{if } n > 2 \end{cases}$$

and $\quad C_n(k) = B_n(k)/\mathrm{Ker}(d_n) \cap \mathrm{Ker}(\tilde{P}_n)$.

Conjecture (Zagier, reformulated by Deligne) For $n \geq 2$:
(i) there is an injection

$$\Psi_n : \frac{\mathrm{Ker}(d_n)}{\mathrm{Ker}(d_n) \cap \mathrm{Ker}(\tilde{P}_n)} \to K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with image a finitely generated group of maximal rank;
(ii) Beilinson's regulator map is given by $(n-1)!\tilde{P}_n$:

$$\begin{array}{ccc} \dfrac{\mathrm{Ker}(d_n)}{\mathrm{Ker}(d_n) \cap \mathrm{Ker}(\tilde{P}_n)} & \xrightarrow{\Psi_n} & K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \\ & \searrow^{(n-1)!\tilde{P}_n} & \Big\downarrow \prod_\sigma reg_\infty \circ \sigma_* \\ & & \mathbb{R}(n-1)^d \end{array}$$

commutes.

Theorem (RdJ; Beilinson-Deligne)
For $n \geq 2$ there exists an injection Ψ_n as in Zagier's conjecture such that the diagram commutes, with finitely generated image.

Remark

- For $n = 2$ this was known before Zagier's conjecture and is due to Bloch and Suslin.
- The image in the theorem has maximal rank for:
  - $\star$ $n = 2$ (Suslin);
  - $\star$ $n = 3$ (Goncharov);
  - $\star$ all $n \geq 2$ if k is cyclotomic; if $\zeta^N = 1$, $\zeta \neq 1$, then $N[\zeta]_n$ lies in Ker(d_n) and the $\Psi_n(N[\zeta]_n)$ generate $K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ (as $\mathbb{Q}$-vector space).

p-adic polylogarithms

Coleman integration on $\mathbb{P}^1_{\mathbb{C}_p}$

- $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$
- $|\cdot|_p$: p-adic valuation with $|p|_p = p^{-1}$
- $\mathcal{O}$: valuation ring
- $\overline{\mathbb{F}}_p$: residue field

Fix a logarithm $\log : \mathbb{C}_p^* \to \mathbb{C}_p$ such that:

- $\log(ab) = \log(a) + \log(b)$;
- $\log(1+z) =$ usual power series expansion for $|z|_p$ small.

For each x in $\mathbb{P}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$ let:

- $U_x =$ residue disc of $x = \{$all y in $\mathbb{P}^1_{\mathbb{C}_p}(\mathbb{C}_p)$ that reduce to $x\}$

[a copy of the maximal ideal of $\mathcal{O}$]

- $t = t_x =$ a local parameter on U_x

[e.g., $t_x = z - \tilde{x}$ if $x \neq \infty$, $t_\infty = 1/z$]

For $x \neq 1, \infty$ in $\mathbb{P}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$ let:

- $A(U_x) = \{\sum_{n=0}^{\infty} a_n t^n$ that converge for $|t|_p < 1\}$
- $A_{\mathrm{log}}(U_x) = A(U_x)$
- $\Omega_{\mathrm{log}}(U_x) = A_{\mathrm{log}}(U_x)dt$

For $x = 1, \infty$ let:

- $A(U_x) = \{\sum_{n=-\infty}^{\infty} a_n t^n$ conv. for $r < |t|_p < 1,$ some $r < 1\}$
- $A_{\mathrm{log}}(U_x) = A(U_x)[\log t]$
- $\Omega_{\mathrm{log}}(U_x) = A_{\mathrm{log}}(U_x)dt$

Then

$$0 \longrightarrow \mathbb{C}_p \longrightarrow A_{\mathrm{log}}(U_x) \xrightarrow{d} \Omega_{\mathrm{log}}(U_x) \longrightarrow 0$$

is exact for each x if we put $d\log(t) = dt/t$.

Theorem (Coleman):
There exists a subspace

$$A_{\text{Col}} \subset \prod_{x \in X(\overline{\mathbb{F}}_p)} A_{\log}(U_x)$$

containing

$$A_{rig} = \lim_{r \uparrow 1} A_{rig}(\mathbb{P}^1_{\mathbb{C}_p} \setminus \{z \text{ such that } |z-1|_p < r \text{ or } |z|_p > 1/r\})$$

and such that, with $\Omega_{\text{Col}} = A_{\text{Col}} dz$,

$$0 \longrightarrow \mathbb{C}_p \longrightarrow A_{\text{Col}} \xrightarrow{d} \Omega_{\text{Col}} \longrightarrow 0$$

is exact.
[$z =$ affine parameter on $\mathbb{A}^1_{\mathbb{C}_p}$]

Definition For ω in Ω_{Col} and P, Q not in U_1 or U_∞, let

$$\int_P^Q \omega = F_\omega(Q) - F_\omega(P)$$

for any F_ω in A_{Col} with $dF_\omega = \omega$.

Example Put $Li_{n+1}(z) = \int_0^z Li_n(y) d\log y$ starting with $Li_0(z) = \frac{z}{1-z}$.

The $Li_n(z)$ are characterized in A_{Col} by

- $Li_n(z) = \sum_{j=1}^\infty \frac{z^j}{j^n}$ for $|z|_p < 1$
- $dLi_{n+1}(z) = Li_n(z)\, d\log(z)$ when $n \geq 0$

- Fact: the $Li_n(z)$ extend to $\mathbb{C}_p \setminus \{1\}$.

The function

$$P_n^p(z) = \sum_{j=0}^{n-1} c_j \log^j(z) Li_{n-j}(z)$$

with $c_0 = 1$ satisfies

$$P_n^p(z) + (-1)^n P_n^p(z^{-1}) = 0$$

if $\sum_{j=0}^{n-1} \frac{c_j}{(n-j)!} = 0$.

We take any such function in what follows.

Theorem (AB-RdJ) For $\sigma : k \longrightarrow F \subset \overline{\mathbb{Q}}_p$ let

$$C_n^\sigma(k, \mathcal{O}) = \langle [x]_n \mid \sigma(x),\ 1 - \sigma(x) \text{ are in } \mathcal{O}^* \rangle$$

$$\subseteq C_n(k) = \frac{B_n(k)}{\mathrm{Ker}(d_n) \cap \mathrm{Ker}(\tilde{P}_n)} .$$

Then $P_n^{p,\sigma} : B_n(k) \to F$ given by $[x]_n \mapsto P_n^p(\sigma(x))$ induces a map $P_n^{p,\sigma} : C_n^\sigma(k, \mathcal{O}) \to F$ and the solid arrows in

$$\begin{array}{ccccc} C_n^\sigma(k,\mathcal{O}) \cap \mathrm{Ker}(d_n) & \hookrightarrow & \dfrac{\mathrm{Ker}(d_n)}{\mathrm{Ker}(d_n) \cap \mathrm{Ker}(\tilde{P}_n)} & \xrightarrow{\Psi_n} & K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \\ & \searrow^{(n-1)! P_n^{p,\sigma}} & & \ddots^{(n-1)! P_n^{p,\sigma}} & \Big\downarrow {\scriptstyle reg_p \circ \sigma_*} \\ & & & & F \end{array}$$

form a commutative diagram.

Remark

- We conjecture that the dotted arrow exists and that the full diagram commutes.
- This holds for $N[\zeta]_n$ in Ker$(d_n)/$Ker$(d_n) \cap$ Ker$(\tilde{P}_n)$ if $\zeta \neq 1$ is an N-th root of unity.

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Proposition

If $\dim_E(E[G]\pi) = 1$ and the (motivic) conjecture applies then:

(1) parts (1)-(3) hold;

(2) part (4) also holds for $\chi_\pi = 1$, $p = 2, \ldots, 19$ and $n = 2, \ldots, 20$;

(3) part (4) also holds for the 470 primitive characters χ_π of $Gal(\mathbb{Q}(\mu_N)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^*$ with $N = 3, \ldots, 49$ for those values of p and n.

Calculations

We checked the conjecture numerically under the assumption that the dotted arrow in the last diagram exists and that the resulting diagram commutes.

We checked the following cases (always for $p = 2, 3, 5, 7, 11$):

- $k/\mathbb{Q}$ a totally real $G = S_3$ extension, π such that $\mathbb{Q}[G]\pi$ is an irreducible 2-dimensional representation of G $(n = 3, 5)$
- $k/\mathbb{Q}$ a totally real $G = D_8$ extension, π such that $\mathbb{Q}[G]\pi$ is an irreducible 2-dimensional representation of G $(n = 3, 5)$
- $k/\mathbb{Q}$ a CM $G = D_8$ extension, π such that $\mathbb{Q}[G]\pi$ is an irreducible 2-dimensional representation of G $(n = 2, 4)$
- $k/\mathbb{Q}$ a totally real $G = S_3 \times \mathbb{Z}/3\mathbb{Z}$ extension, π such that $\mathbb{Q}(\zeta_3)[G]\pi$ is an irreducible 2-dimensional representation of S_3 multiplied by a non-trivial character of $\mathbb{Z}/3\mathbb{Z}$ $(n = 3, 5)$

The conjecture holds numerically in all cases considered, with $e(n,k)/e_p(n,k) = 1 + O(p^{M(p)})$ where $M(2) = 72$, $M(3) = 47$, $M(5) = 32$, $M(7) = 26$ and $M(11) = 22$ in the first three cases. It also holds in the last, where we took $M(p)$ such that $p^{M(p)}$ was about 10^{16}.

Some values

Splitting field of $x^6 - 3x^5 - 2x^4 + 9x^3 - 5x + 1$ $G = S_3$ $n = 3$

p	$R_{3,p}(M_\pi^{\mathbb{Q}})/D(M_\pi^{\mathbb{Q}})^{1/2,p}$
2	$(1.0001010100100000000100001011111001\cdots)_2 \times 2^0$
3	$(1.0100121212022110212000001102121121\cdots)_3 \times 3^4$
5	$(4.2320143330214023104113344110103 13\cdots)_5 \times 5^6$
7	$(6.354304301412412415450326016336635)_7 \times 7^6$
11	$(2.62161235A928A3423563A7888A)_{11} \times 11^6$

p	$L_p(3, \chi_\pi \otimes \omega_p^{-2}, \mathbb{Q})$
2	$(1.0000001010011100010000011001000111\cdots)_2 \times 2^2$
3	$(1.1202122002221002110112100012002 00\cdots)_3 \times 3^0$
5	$(4.1104002444023244223302413104001 41\cdots)_5 \times 5^0$
7	$(5.235163632265012613625435331 10)_7 \times 7^0$
11	$(A.9542728A692401225487A278)_{11} \times 11^0$

Splitting field of $x^4 - 2x^3 + 5x^2 - 4x + 2$ $G = D_8$ $n = 4$

p	$R_{4,p}(M_\pi^{\mathbb{Q}})/D(M_\pi^{\mathbb{Q}})^{1/2,p}$
2	$(1.01111010101011110100100110010011 0\cdots)_2 \times 2^{15}$
3	$(2.020202022222222120111210002222000\cdots)_3 \times 3^{10}$
5	$(3.121141434441031100040041101320123\cdots)_5 \times 5^{8}$
7	$(4.301032463651051654014512630106 36)_7 \times 7^{8}$
11	$(9.886432141481592 8A83132426)_{11} \times 11^{8}$

p	$L_p(4, \chi_\pi \otimes \omega_p^{-3}, \mathbb{Q})$
2	$(1.111000011111001011100110001010000\cdots)_2 \times 2^{6}$
3	$(1.211021010020211100000111212012102\cdots)_3 \times 3^{0}$
5	$(1.224142013121421301311123344301232\cdots)_5 \times 5^{1}$
7	$(6.662006052166436554346006064 23)_7 \times 7^{0}$
11	$(3.5059939560A2AA91106844A7)_{11} \times 11^{0}$