

Lower bounds for the measurable chromatic number of the Euclidean space

Fernando Mário de Oliveira Filho, CWI

Joint work with:

Christine Bachoc, Université Bordeaux

Gabriele Nebe, RWTH Aachen

Frank Vallentin, CWI

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The chromatic number of \mathbb{R}^n

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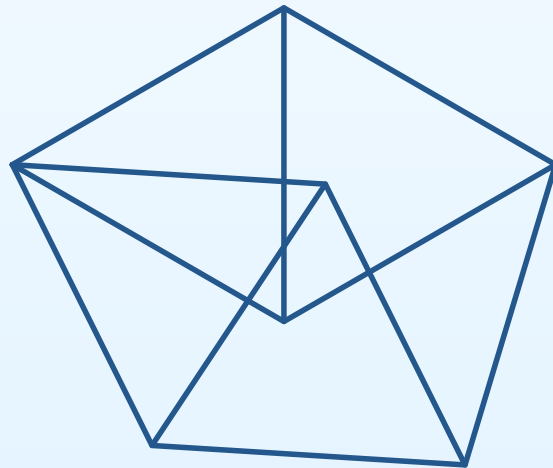
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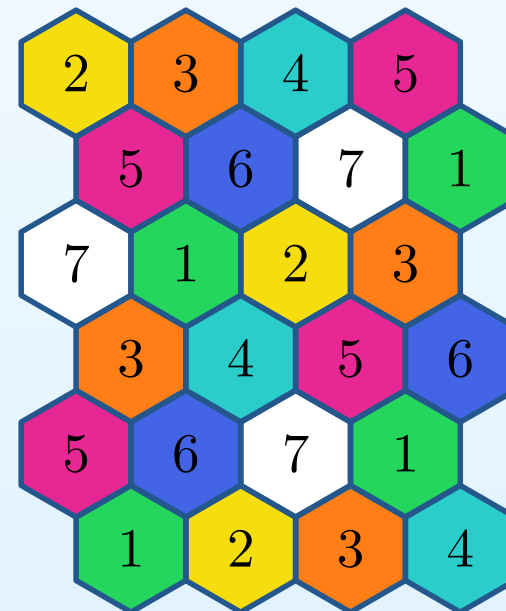
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Moser graph



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- De Bruijn and Erdős (1951): $\chi(\mathbb{R}^n)$ is attained by a finite subgraph. Proof uses the axiom of choice.
- **Observation:** $\chi(\mathbb{R}^n)$:= chromatic number of graph with $V := \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$ adjacent if $\|x - y\| = 1$.

The measurable chromatic number

Definition: a set $C \subseteq \mathbb{R}^n$ is **stable** if $\|x - y\| \neq 1$ for all $x, y \in C$.

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- Székely and Wormald (1989): table of lower bounds for $\chi_m(\mathbb{R}^n)$ for $n \leq 24$.
- **Our results:** improved bounds for $\chi_m(\mathbb{R}^n)$ for $n \geq 10$ based on a generalization of the Lovász's theta function for finite graphs.

A table of improvements

New improved lower bounds for $\chi_m(\mathbb{R}^n)$, compared to best previous lower bounds.

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A related problem

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Then: $\chi_m(G(n, t)) \leq \chi_m(\mathbb{R}^n)$.

The stability number

For $-1 < t < 1$ define the **stability number** of $G(n, t)$:

$$\alpha(G(n, t)) := \sup\{\omega(C) \mid C \subseteq S^{n-1} \text{ measurable and stable}\}.$$

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So an upper bound α' for α gives a lower bound for χ_m :

$$\frac{\omega_n}{\alpha'(G(n, t))} \leq \frac{\omega_n}{\alpha(G(n, t))} \leq \chi_m(G(n, t)).$$

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- **invariant**, if for all $A \in O(\mathbb{R}^n)$ and all $x, y \in S^{n-1}$,

$$K(Ax, Ay) = K(x, y).$$

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For our infinite case:

$$\vartheta(G(n, t)) := \sup \int_{S^{n-1}} \int_{S^{n-1}} K(x, y) d\omega(x) d\omega(y)$$
$$K(x, x) = 1/\omega_n \quad \text{for all } x \in S^{n-1},$$
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$C \subseteq S^{n-1}$ a measurable stable set. $\chi^C: S^{n-1} \rightarrow \{0, 1\}$ its characteristic function. Note $\chi^C \in L^2(S^{n-1})$.

A generalization of the Lovász theta function (cont.)

$C \subseteq S^{n-1}$ a measurable stable set. Consider kernel K such that

$$K(x, y) = \frac{1}{\omega(C)} \int_{O(\mathbb{R}^n)} \chi^C(Ax) \chi^C(Ay) d\mu(A).$$

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To fix: approximate χ^C by good continuous functions using the regularity of ω and Urysohn's lemma.

Schoenberg's representation theorem

Theorem (Schoenberg 1941). A continuous kernel $K \in L^2(S^{n-1} \times S^{n-1})$ is invariant and positive **if and only if** there exist $f_0 \geq 0, f_1 \geq 0, \dots$ such that

$$K(x, y) = \sum_{k=0}^{\infty} f_k P_k^n(x \cdot y)$$

for all $x, y \in S^{n-1}$, with **absolute and uniform** convergence.

$P_k^n(u)$ is the Gegenbauer polynomial of degree k and parameter

$\lambda := (n - 2)/2$, normalized so that $P_k^n(1) = 1$.

Schoenberg's representation theorem (cont.)

Primal problem

$\vartheta(G(n, t))$ is equal to:

$$\begin{aligned} \sup \quad & \omega_n^2 f_0 \\ & f_k \geq 0 \text{ for } k = 0, 1, \dots, \\ & \sum_{k=0}^{\infty} f_k P_k^n(1) = 1/\omega_n, \\ & \sum_{k=0}^{\infty} f_k P_k^n(t) = 0. \end{aligned}$$

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$$\begin{aligned} \leq \quad & \inf \quad z_1/\omega_n \\ & z_1 P_0^n(1) + z_t P_0^n(t) \geq \omega_n^2, \\ & z_1 P_k^n(1) + z_t P_k^n(t) \geq 0 \\ & \text{for } k = 1, 2, \dots \end{aligned}$$

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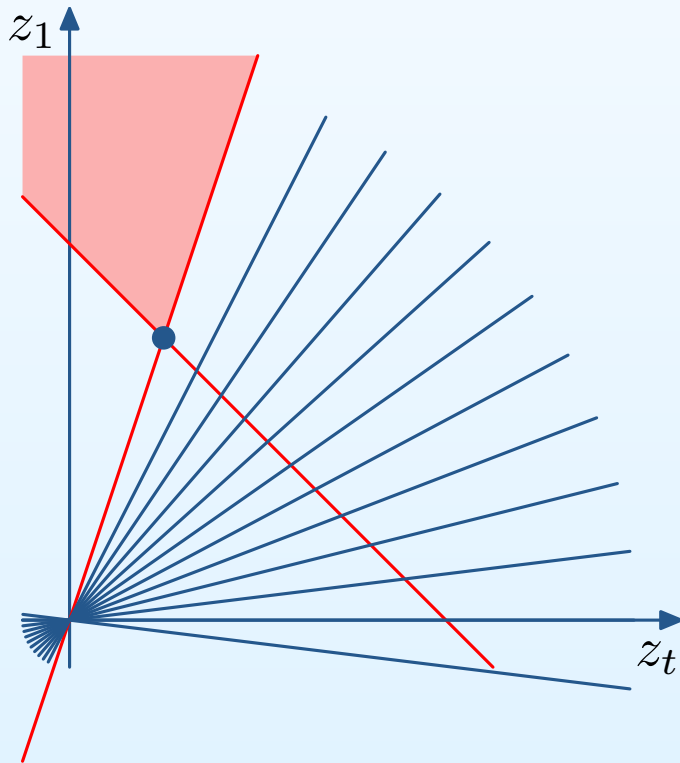
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Simplification:

$$\begin{aligned} \inf \quad & z_1/\omega_n \\ & z_1 + z_t \geq \omega_n^2, \\ & z_1 + z_t P_k^n(t) \geq 0 \\ & \text{for } k = 1, 2, \dots \end{aligned}$$

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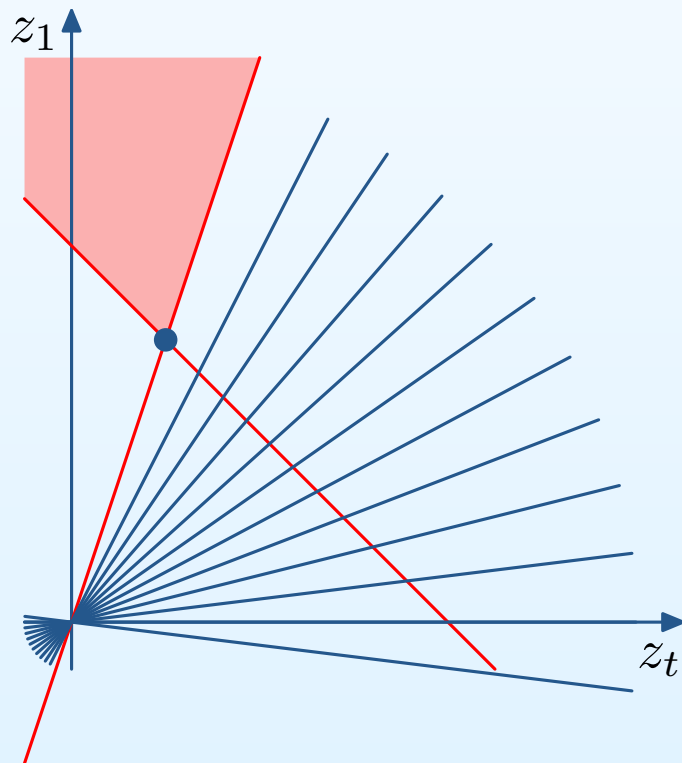


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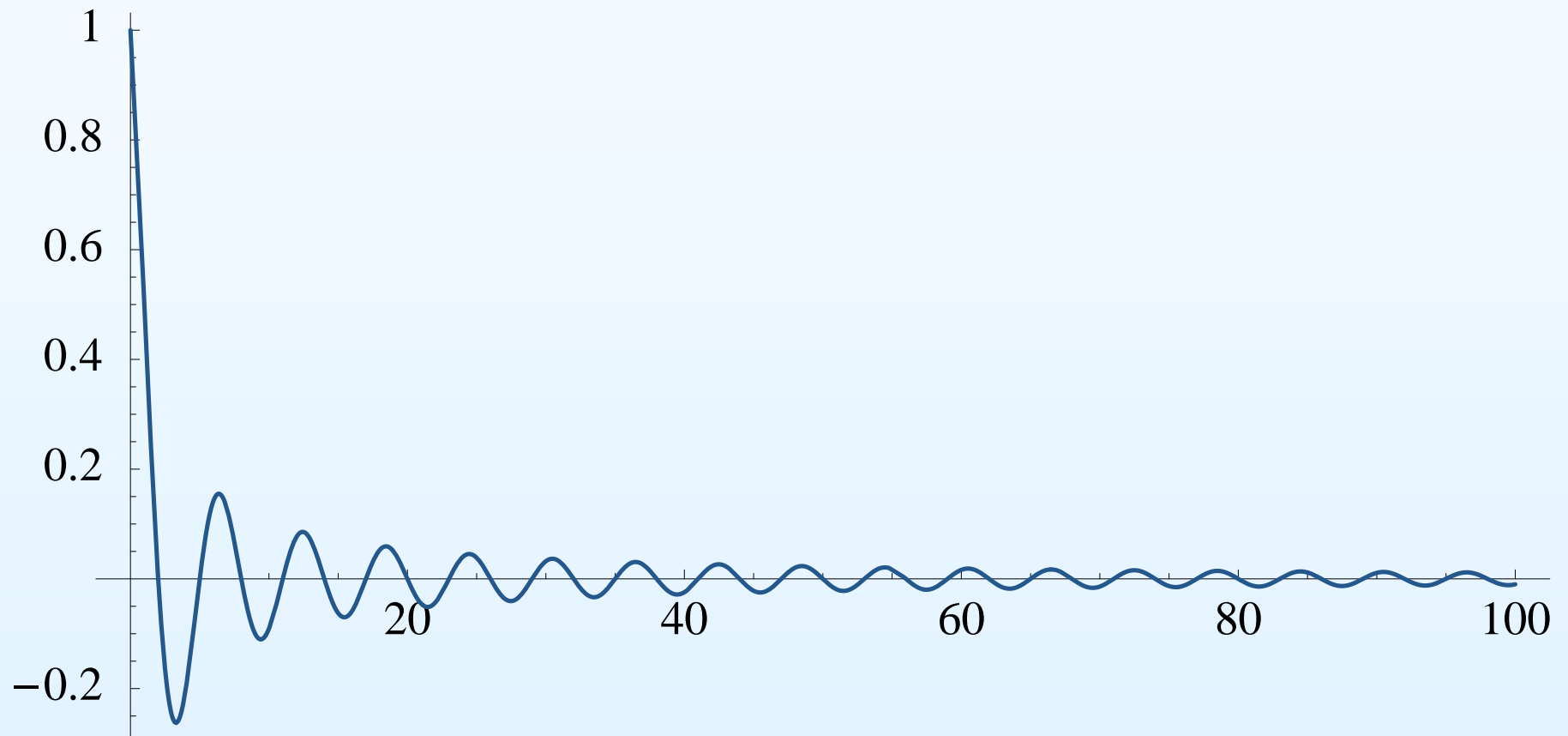


- $m := \min_{k \geq 0} P_k^n(t);$
- $\vartheta(G(n, t)) = \omega_n \frac{m}{m - 1};$
- $\chi_m(G(n, t)) \geq \frac{\omega_n}{\vartheta(G(n, t))} = \frac{m - 1}{m}.$

Solving the dual (cont.)

It suffices to check finitely many degrees, as

$$\lim_{k \rightarrow \infty} |P_k^n(t)| = 0.$$



A table of improvements

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