Examples.

K denotes an algebraically closed field of characteristic 0 and $K(z)$ is provided with the differentiation $\frac{d}{dz}$.

(1) *Let f be a non zero solution of the equation $y' = y$ over $K(z)$, then f is transcendental over $K(z)$.* In particular, *e^z is transcendental over $\mathbb{C}(z)$.*

Proof: Let $f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 = 0$ be the minimal equation over $K(z)$. Differentiate: $f^n + \frac{a'_{n-1}+(n-1)a_{n-1}}{n} f^{n-1} + \cdots + \frac{a'_1+a_1}{n} f + \frac{a'_0}{n} = 0$. Hence $a'_0 = na_0$ and $a_0 \in K(z),\ a_0 \neq 0$ yield a contradiction.

(2) *Given are* $Q_1, \ldots, Q_n \in K(z)$*. Let* $f_1, \ldots, f_n$ *be non zero solutions of the equations* $y_i' = Q_i y_i$*. Then the* $\{f_i\}$ *are algebraically dependent over* $K(z)$ *if and only if for some integers* m_i*, not all zero, such that the equation*
$y' = (\sum m_i Q_i)y$ *has a solution in* $K(z)^*$*.*
In particular, $e^{P_1}, \ldots, e^{P_n}$*, with all* $P_i \in \mathbb{C}(z) \setminus \mathbb{C}$*, are algebraically dependent over* $\mathbb{C}(z)$ *if and only if there are integers* m_i*, not all zero, such that* $\sum m_i P_i \in \mathbb{C}$*.*

Proof: Suppose that the monomials $f_1^{m_1} \cdots f_n^{m_n}$ with $\underline{m} := (m_1, \ldots, m_n) \in \mathbb{Z}^n$ are linearly dependent over $K(z)$. Consider a non trivial expression $\sum_{\underline{m}} c(\underline{m}) f_1^{m_1} \cdots f_n^{m_n} = 0$, with all $c(\underline{m}) \in K(z)$, which has a support S of minimal cardinality. One may suppose that $c(\underline{0}) = 1$. Differentiation yields an identity with smaller support. Hence for $\underline{m} \in S$, $\underline{m} \neq \underline{0}$ one has $c(\underline{m})' + c(\underline{m}) \cdot (\sum m_i Q_i) = 0$.

In the special case $Q_i = P_i'$, the expression $Q := -\sum m_i P_i'$ is a derivative and moreover $Q = \frac{c(\underline{m})'}{c(\underline{m})}$ has the form $\sum \frac{n_i}{z-a_i}$ with $n_i \in \mathbb{Z}$ and $a_i \in \mathbb{C}$. This is only possible for $Q = 0$.

(3) *Any solution f of $y' = z^{-1}$ is transcendental over $K(z)$.* In particular, $\log z$ *transcendental over* $\mathbb{C}(z)$.

Proof: Let $f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = 0$ be the minimal equation of f over $K(z)$. Differentiation produces an equation of lower degree $nz^{-1} f^{n-1} + a_{n-1}' f^{n-1} + a_{n-1} z^{-1}(n-1) f^{n-2} + \cdots + a_1' f + a_1 z^{-1} + a_0' = 0$. This yields the contradiction $nz^{-1} + a_{n-1}' = 0$.

(4) $\log P_1, \ldots, \log P_n$ *with all* $P_i \in \mathbb{C}(z) \setminus \{0\}$ *are algebraically independent if and only if the* $\frac{P_i'}{P_i}$ *are linearly independent over* $\mathbb{C}$.

Let $\mathbb{K}$ be a differential field with field of constants $C \neq \mathbb{K}$ (which is algebraically closed and has characteristic 0). A system of linear differential equations over $\mathbb{K}$ has the form $y' = Ay$ where A is a $n \times n$ matrix with entries in $\mathbb{K}$. The *Picard-Vessiot ring* $R = PVR(A)$ for the equation is a differential extension of $\mathbb{K}$ having the properties:

(1) there exists an invertible matrix F, with entries in R such that $F' = AF$. (F is called a fundamental matrix).

(2) the only differential ideals of R are $\{0\}$ and R.

(3) R is generated as $\mathbb{K}$-algebra by the entries of F and $\frac{1}{\det F}$.

The construction of R is as follows. Start with the differential ring $R_0 := \mathbb{K}[\{X_{i,j}\}, \frac{1}{\det(X_{i,j})}]$ with differentiation given by $(X'_{i,j}) = A \cdot (X_{i,j})$. Choose a differential ideal $I \subset R_0$, maximal among the set of all differential ideals. Then $R \cong R_0/I$.

Some theorems.

1. I is a prime ideal. The field of fractions of $R = PVR(A)$ is called the Picard-Vessiot field $PVF(A)$.

2. The differential Galois group $G = dGal(A)$ is the group of the differential automorphisms of $R/\mathbb{K}$. Any $\sigma \in G$ acts on F and $\sigma(F) = FC(\sigma)^{-1}$ with $C(\sigma) \in \mathrm{GL}(n, C)$.

The map $\sigma \mapsto C(\sigma)$ embeds G as a *linear algebraic group* in $\mathrm{GL}(n, C)$.

3. Let $J \subset R_0$ be a differential ideal. One considers the group $G(J) \subset \mathrm{GL}(n, C)$ consisting of the elements M such that the $\mathbb{K}$-automorphism of R_0, given by $(X_{i,j}) \mapsto (X_{i,j})M$, leaves the ideal J invariant. Then $G(J)$ is an algebraic subgroup. Let $I \supset J$ be a maximal differential ideal, then $G(I) \subset G(J)$ and $G(I) = dGal(A)$.

4. Suppose that there is given some differential field $\mathbb{L} \supset \mathbb{K}$ with field of constants C such that there exists a matrix F with entries in $\mathbb{L}$, $\det F \neq 0$ and $F' = AF$. Then the subalgebra $\mathbb{K}[\text{entries of } F, \frac{1}{\det F}]$ of $\mathbb{L}$ is the Picard-Vessiot ring $PVR(A)$.

5. The dimension of $dGal(A)$ is equal to the transcendence degree of $PVF(A)$ over $\mathbb{K}$.

6. There is a Galois correspondence between the algebraic subgroups $H \subset G$ and the intermediate differential fields $\mathbb{L}$ with $\mathbb{K} \subset \mathbb{L} \subset PVF(A)$. This is given by $H \mapsto PVF(A)^H$ and $\mathbb{L} \mapsto dGal(PVF(A)/\mathbb{L})$.
   In particular $PVF(A)^{dGal(A)} = \mathbb{K}$.

Again examples:

(i) *Equation* $y' = ay$ *with* $a \in \mathbb{K}^*$.
The Picard-Vessiot ring has the form $\mathbb{K}[x, x^{-1}] := \mathbb{K}[X, X^{-1}]/I$ with $X' = aX$ and I a maximal differential ideal. The group $G(\{0\}) = C^*$ acts by $X \mapsto cX$. If $I = \{0\}$, then the differential Galois group is the multiplicative group $\mathbb{G}_m = C^*$.

If $I \neq \{0\}$, then the differential Galois group is the cyclic group $\mu_m \subset C^*$ (for some $m \geq 1$) of the mth roots of unity. A generator σ of this group acts by $\sigma x = e^{2\pi i/m} x$. In particular, $\sigma x^m = x^m$ and thus $x^m = g \in \mathbb{K}^*$. It follows that $I = (X^m - g)$ and that $g' = m \cdot ag$.

(ii) *Equations* $y_i' = a_i y_i$ *for* $i = 1, \ldots, n$ *and all* $a_i \in \mathbb{K}^*$*.*

In matrix form $y' = diag(a_1, \ldots, a_n)y$. The algebra $R_0 = \mathbb{K}[\{X_{i,j}\}, \frac{1}{\text{det}}]$ has the differential ideal $(\{X_{i,j} |\ i \neq j\})$. After dividing this out the Picard-Vessiot ring is $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] := \mathbb{K}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]/I$ with $X_i' = a_i X_i$ for all i and I a maximal differential ideal. If $I = 0$, then the differential Galois group is $\mathbb{G}_m^n$. If $I \neq \{0\}$ the the differential Galois group G is a proper subgroup of $\mathbb{G}_m^n$ and is a subgroup of $\{(t_1, \ldots, t_n) \in \mathbb{G}_m^n |\ t_1^{m_1} \cdots t_n^{m_n} = 1\}$ for a suitable non zero tuple $(m_1, \ldots, m_n) \in \mathbb{Z}^n$.

Thus any $(t_1, \ldots, t_n) \in G$ leaves the element $x_1^{m_1} \cdots x_n^{m_n}$ invariant. By Galois correspondence, this element belongs to $\mathbb{K}^*$. In other words, I contains an element of the form $X_1^{m_1} \cdots X_n^{m_n} - g$ for some $g \in \mathbb{K}^*$.

(iii) *Again the example* $y_i = e^{P_i}$ *for* $i = 1, \ldots, n$. This corresponds to the equations $y_i' = P_i' y_i$. Let $\mathcal{M}(U)$ be the field of meromorphic functions on some connected open set $U \subset \mathbb{C}$, not containing the (possible) poles of the P_i. Consider the $\mathbb{C}(z)$-algebra homomorphism $\mathbb{C}(z)[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \to \mathcal{M}(U)$, given by $X_i \mapsto e^{P_i}$. The image of this map is the Picard-Vessiot ring and the kernel I of this map is a maximal differential ideal. If the e^{P_i} are algebraically dependent over $\mathbb{C}(z)$, then $I \neq \{0\}$ and thus I contains an element of the form $X_1^{m_1} \cdots X_n^{m_n} - g$ for some $g \in \mathbb{C}(z)^*$. Thus $g = y_1^{m_1} \cdots y_n^{m_n}$ satisfies $\frac{g'}{g} = (\sum m_i P_i)'$. This implies, as before, $\sum m_i P_i \in \mathbb{C}$.

(iv) *Now we investigate additive equations.*
Consider a system of equations $y_i = a_i$, $i = 1, \ldots, n$ with all $a_i \in \mathbb{K}$. The Picard-Vessiot ring is seen to have the form $\mathbb{K}[X_1, \ldots, X_n]/I$, with $X_i' = a_i$ and where I is a maximal differential ideal. Now $I = \{0\}$ is equivalent to: the solutions $y_1, \ldots, y_n$ (in a differential field $\mathbb{L} \supset \mathbb{K}$) are algebraically independent over $\mathbb{K}$. In this case the differential Galois group is $C^n = \mathbb{G}_a^n$. If $I \neq \{0\}$, then the differential Galois group G is a proper algebraic subgroup of $\mathbb{G}_a^n$ and lies in $\{(t_1, \ldots, t_n) \in \mathbb{G}_a^n | \ \lambda_1 t_1 + \cdots + \lambda_n t_n = 0\}$ for some $(\lambda_1, \ldots, \lambda_n) \in C^n \setminus \{0\}$. Hence any element in G will leave the element $\sum_i \lambda_i y_i$ invariant and thus $g := \sum_i \lambda_i y_i \in \mathbb{K}$. In other words, the equation $y' = \sum \lambda_i a_i$ has a solution in $\mathbb{K}$.

(v) *Again the example* $y_i = \log P_i,\ i = 1, \ldots, n$. One finds the equations $y_i' = \frac{P_i'}{P_i}$. And the algebraic dependence of the y_i is equivalent to: some non trivial $\mathbb{C}$-linear combination $\sum \lambda_i \frac{P_i'}{P_i}$ is a derivative of some element $g \in \mathbb{C}(z)$. One observes that $\sum \lambda_i \frac{P_i'}{P_i}$ has the form $\sum \frac{c_i}{z-a_i}$ and this is a derivative only if all $c_i = 0$. In other words the $\frac{P_i'}{P_i}$ are linearly dependent over $\mathbb{C}$.

(vi) *Final example.*

Consider a system of equations $y_i' = a_i y_i,\ i = 1, \dots, a$ and $z_j' = b_j,\ j = 1, \dots, b$. The differential Galois group G of this system is an algebraic subgroup of $\mathbb{G}_m^a \times \mathbb{G}_a^b$. If G is the whole group, then the $y_1, \dots, z_b$ are algebraically independent over $\mathbb{K}$. In the opposite case, there is a non trivial relation $\sum m_i a_i = \frac{g'}{g}$ for some $g \in \mathbb{K}^*$ and $m_i \in \mathbb{Z}$ or there is a non trivial relation $\sum \lambda_j b_j = g'$ for some $g \in \mathbb{K}$ and $\lambda_i \in C$.

Indeed, any algebraic subgroup of $\mathbb{G}_m^a \times \mathbb{G}_a^b$ is a product of an algebraic subgroup of $\mathbb{G}_m^a$ and one of $\mathbb{G}_a^b$.

Some difference Galois theory.

The field K is supposed to be algebraically closed and to have characteristic 0. On $\mathbb{K} = K(z)$ one considers the difference operator ϕ given by $\phi(f)(z) = f(z+1)$. A system of linear difference equations can be put into the form $\phi(y) = Ay$ with A an invertible matrix with entries in $\mathbb{K}$. As before one can define and construct the Picard-Vessiot ring for a difference equation and introduce the difference Galois group. There is a small difference with the former construction, namely the PVR can have zero divisors but has however no nilpotent elements. In the formulation of the Galois correspondence one has to replace the field of fractions of PVR by its total quotient ring.

Basic example for this: $\phi(y) = -y$.
Then $PVR = \mathbb{K}[x, x^{-1}] := \mathbb{K}[X, X^{-1}]/(X^2 - 1)$ with ϕ defined by $\phi(x) = -x$.

$\phi(y) - y = a \in K(z)$ has a solution in $K(z)$ if and only if a has the form $p(z) + \sum_{m \geq 1,\ \alpha \in K} \frac{c(m,\alpha)}{(z-\alpha)^m}$ where $p(z)$ is a polynomial and for every $m \geq 1$ and every $A \in K/\mathbb{Z}$ one has $\sum_{\alpha \in A} c(m, \alpha) = 0$.

$\phi(y) = ay$ with $a \in K(z)^*$ has a non zero solution in $K(z)$ if and only if a has the form $\prod (z-\alpha)^{m(\alpha)}$ such that for every $A \in K/\mathbb{Z}$ one has $\sum_{\alpha \in A} m(\alpha) = 0$.

Applications.

(1) $f = e^z$ *is transcendental over* $\mathbb{C}(z)$.
Proof: f is a non zero solution of $\phi(y) = e{\cdot}y$. If f is algebraic, then the difference Galois group is a proper algebraic subgroup μ_m of $\mathbb{G}_m$. However $\phi(g) = e^m g$ has no solution in $\mathbb{C}(z)^*$.

(2) *The Gamma function is hyper transcendental, i.e., the collection* $\{\Gamma^{(m)} |\ m \geq 0\}$ *is algebraically independent over* $\mathbb{C}(z)$.
Even stronger:
Let K *denote (the algebraic closure of) the field of meromorphic 1-periodic functions on* $\mathbb{C}$ *(i.e.,* $h(z+1) = h(z)$*). Then the functions* $\{\Gamma^{(m)} |\ m \geq 0\}$ *are algebraically independent over* $K(z)$.

Proof: Γ is a non zero solution of $\phi(y_0) = z \cdot y_0$. Further $\frac{\Gamma'}{\Gamma}$ is a solution of $\phi(z_0) - z_0 = z^{-1}$ and $(\frac{\Gamma'}{\Gamma})^{(m)}$ is a solution of $\phi(z_m) - z_m = (z^{-1})^{(m)}$. We consider the equations $\phi(y_0) = zy_0$ and $\phi(z_m) - z_m = (\frac{1}{z})^{(m)}$ for $m = 0, \ldots, N$ over the field $K(z)$. Hyper transcendence will follow from the statement that the difference galois group of this set of equations is $\mathbb{G}_m \times \mathbb{G}_a^{N+1}$. Thus (as in the differential case) we have to prove that $\phi(g) = z^m g$ (with $m \in \mathbb{N}$) has no solution in $K(z)^*$ and that $\phi(g) - g = \sum_{i=1}^{N+1} \lambda_i z^{-i}$ (not all $\lambda_i \in K$ are zero) has no solution in $K(z)$. Both statements follow from the above criteria. $\square$

Generalizations, theorems of Ch. Hardouin.

Theorem (A). One associates to $a \in \mathbb{C}(z)^*$ the 'divisor' $D : \mathbb{C}/\mathbb{Z} \to \mathbb{Z}$, defined by $D(A) = \sum_{\alpha \in A} ord_\alpha(a)$. Let elements $a_1, \ldots, a_n \in \mathbb{C}(z)^*$ be given such that *their divisors* $D_1, \ldots, D_n : \mathbb{C}/\mathbb{Z} \to \mathbb{Z}$ *are linearly independent over* $\mathbb{Z}$.

Consider the system of difference equations $\phi(Y_i) = a_i Y_i$ for $i = 1, \ldots, n$ and $\phi(Z_{i,m}) = Z_{i,m} + (\frac{d}{dz})^m (a_i^{-1} \frac{da_i}{dz})$ for $i = 1, \ldots, n$; $0 \leq m \leq N$. Then the difference Galois group of this system is $\mathbb{G}_m^n \times \mathbb{G}_a^{n(N+1)}$.

In particular, suppose that the non zero analytic functions $f_1, \ldots, f_n$ *satisfy* $\phi(f_i) = a_i f_i$ *all* i. *Then* $\{f_1^{(m_1)} \cdots f_n^{(m_n)} |\ m_1, \ldots, m_n \geq 0\}$ *is algebraically independent over* $K(z)$, *where* K *is the field of the 1-periodic meromorphic functions on* $\mathbb{C}$.

Theorem (B). The difference operator $f(z) \mapsto f(z+1)$ and the differential operator $f \mapsto \frac{d}{dz}f$ are now replaced by $f(z) \mapsto f(qz)$ (say with $0 < |q| < 1$) and $f \mapsto z\frac{d}{dz}f$. One associates to $a \in \mathbb{C}(z)^*$ the divisor $D : \mathbb{C}^*/q^{\mathbb{Z}} \to \mathbb{Z}$, given by $D(A) = \sum_{\alpha \in A} ord_\alpha(a)$.

A similar result holds in this case:
Let the non zero analytic functions $f_1, \ldots, f_n$ *satisfy* $f_i(qz) = a_i f(z)$ *for* $i = 1, \ldots, n$. *Suppose that the divisors of the* a_i *are independent over* $\mathbb{Z}$. *Then* $\{f_1^{(m_1)} \cdots f_n^{(m_n)} |\ m_1, \ldots, m_n \geq 0\}$ *are algebraically independent over* $\mathbb{C}(E)(z)$, *where* E *is the elliptic curve* $\mathbb{C}^*/q^{\mathbb{Z}}$ *and* $\mathbb{C}(E)$ *denotes the field of the meromorphic functions on* E.

In theorem (A) one can interchange the roles of $\frac{d}{dz}$ and ϕ.

Theorem (C). *Suppose that the solution* f *of the equation* $f' = a \in \mathbb{C}(z)$ *is not a rational function. Then the family* $\{\phi^n(f) |\ n \in \mathbb{Z}\}$ *is algebraically independent over* $\mathbb{C}(z)$*.*
In particular, *the family* $\{\log(z-n) |\ n \in \mathbb{Z}\}$ *is algebraically independent over* $\mathbb{C}(z)$*.*

Proof. One associates to any $b \in \mathbb{C}(z)$ the 'divisor' $R(b) : \mathbb{C} \to \mathbb{C}$ given by $R(b)(\alpha) = Res_\alpha(b\ dz)$. The equation $y' = b$ has a rational solution if and only if $R(b) = 0$.

It is given that $R(a) \neq 0$. Then also, for any non trivial $\mathbb{C}$-linear combination $b = \sum \lambda_i \phi^i(a)$ on has $R(b) \neq 0$. $\square$

Likewise one can interchange the roles of ϕ and $z\frac{d}{dz}$ in (B) and one obtains for instance:

Let f *satisfy* $z\frac{d}{dz}f = a \in \mathbb{C}(z)$*. If* f *is not rational, then the functions* $\{f(q^n z)|\ n \in \mathbb{Z}\}$ *are algebraically independent over* $\mathbb{C}(E)(z)$*.*

It seems also possible to derive results on the transcendence of functions in more variables, like the Beta function $B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$, by differential and difference Galois theory equations in two variables (or more).