

Faster arithmetic on elliptic curves – blessing to ECC, harm to RSA

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**It all started with the fight
between elliptic and
hyperelliptic curves . . .**

To face the challenge, to take the competition to a completely new level

...

$$y^2 = x^3 + ax + b$$

... elliptic has to reconsider its form ...

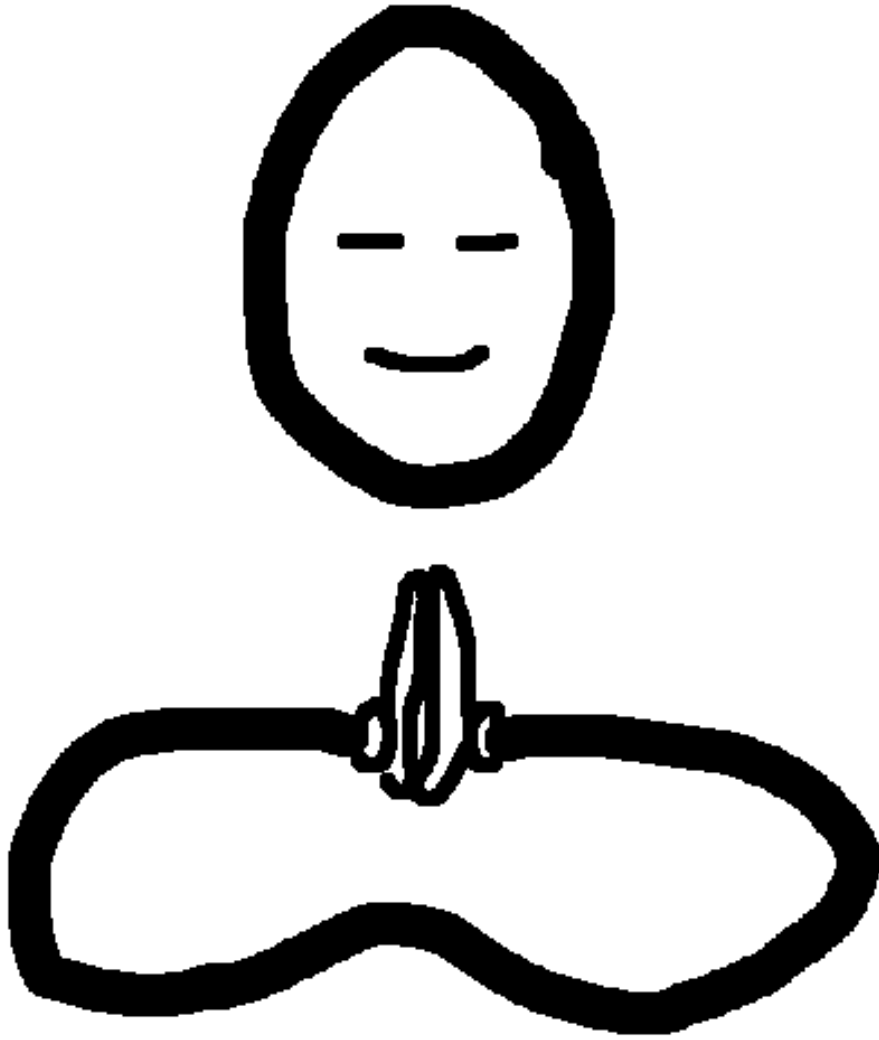


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Elliptic Curves – p. 4

... has to abstract from its Weierstrass form

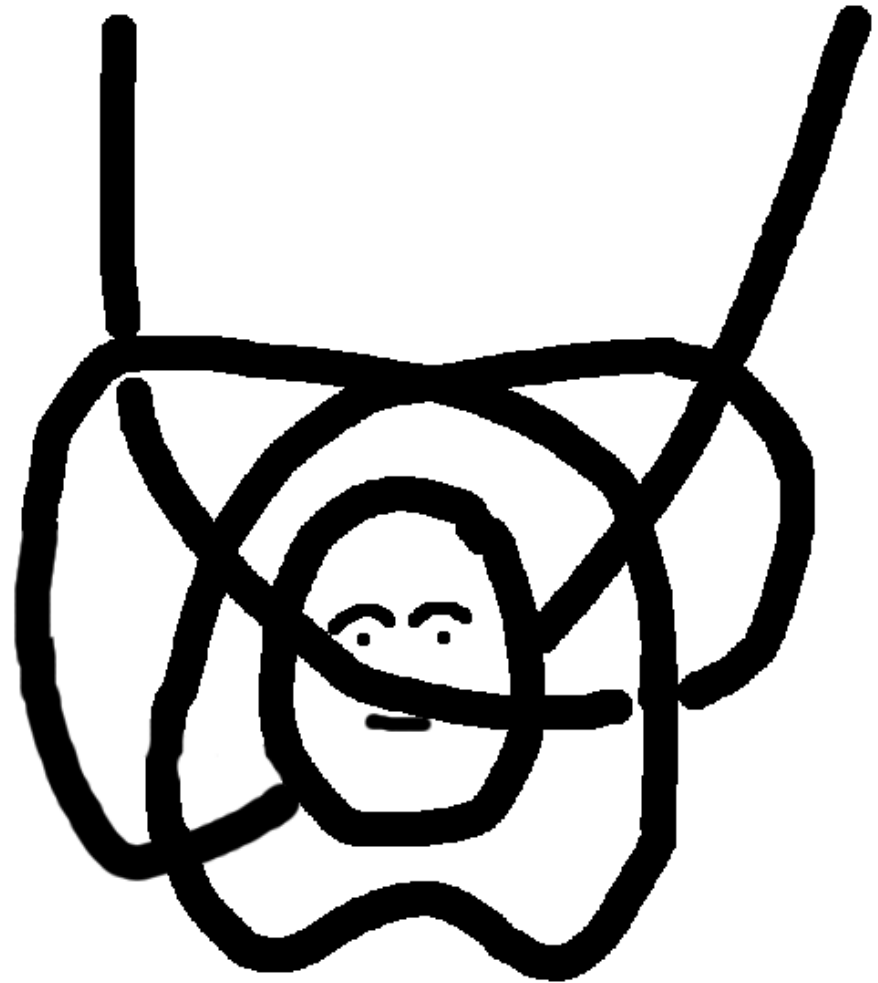


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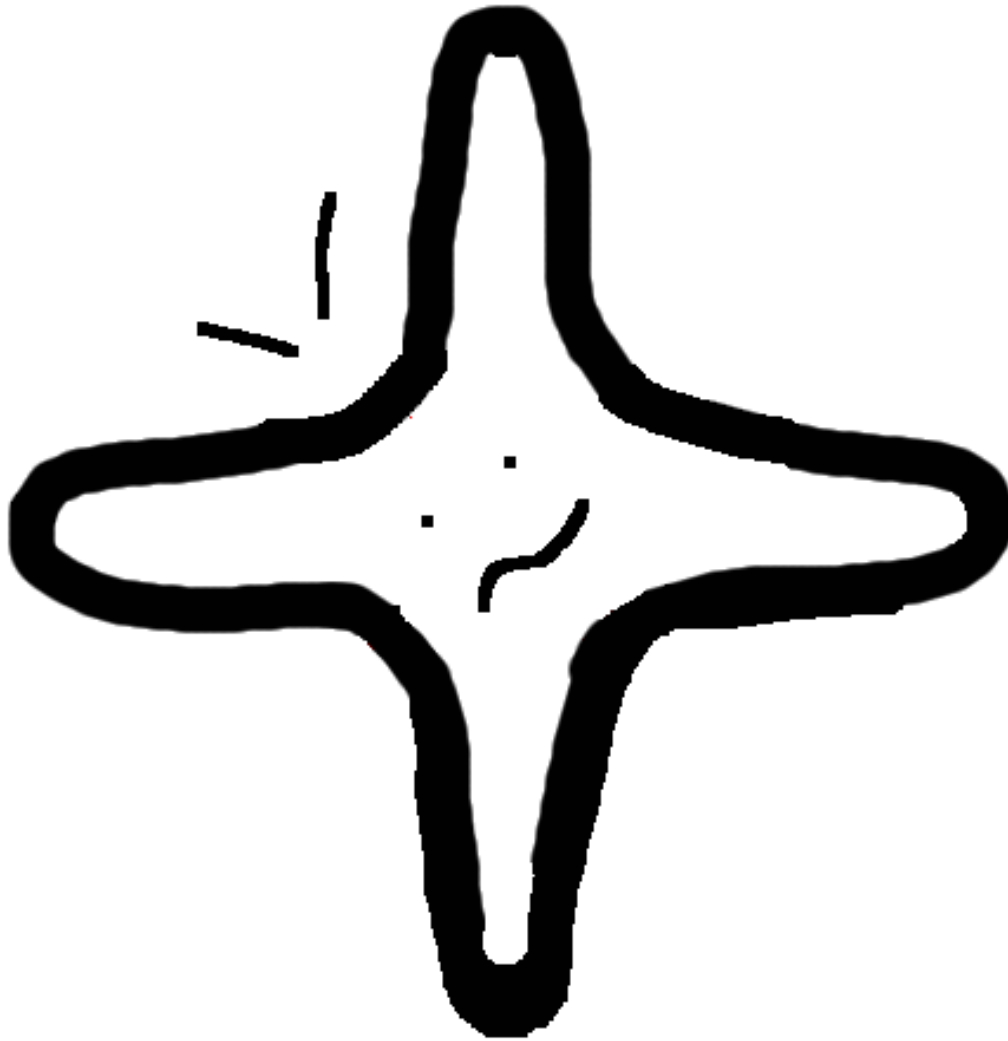
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Elliptic Curves – p. 5

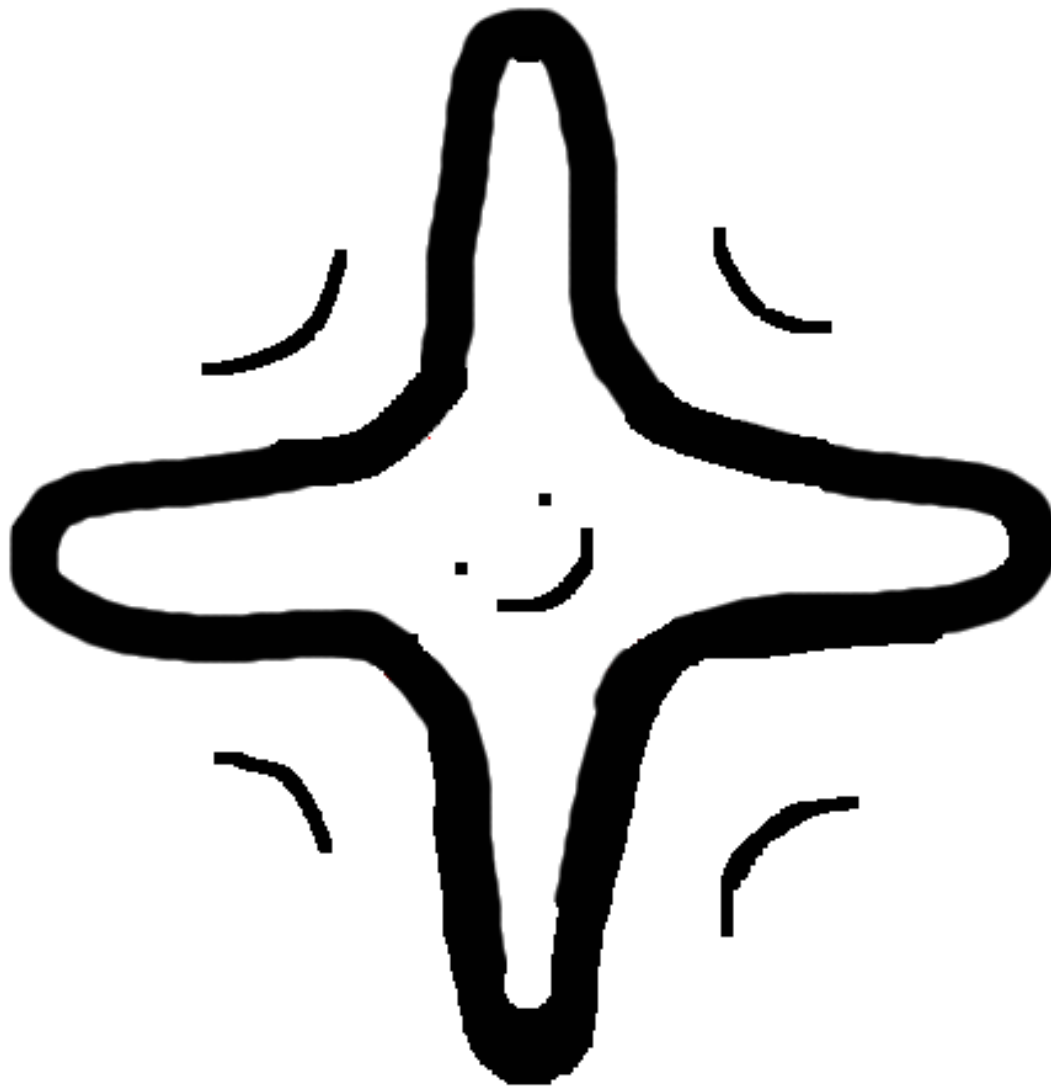
... has to undergo severe isomorphic transformations ...



... until it finds ...



...its true ...



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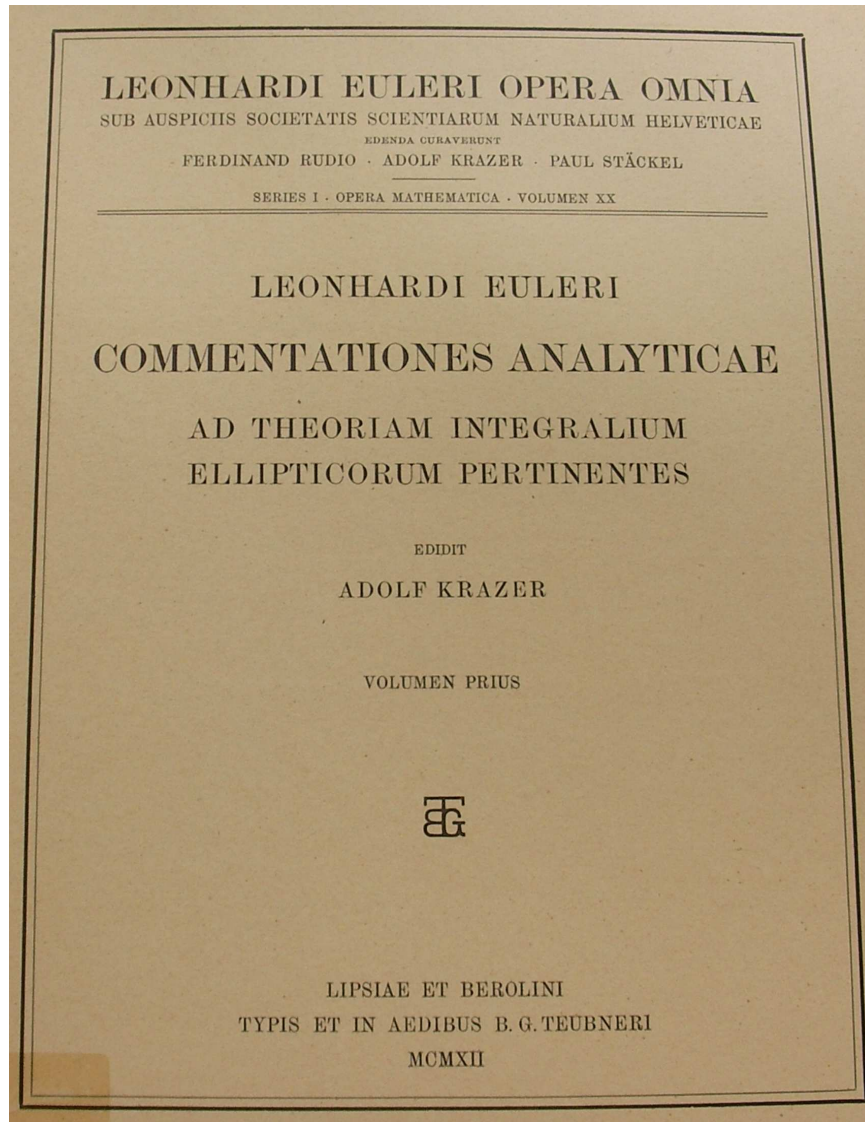
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Elliptic Curves – p. 8

... normal form!

$$X^2 + Y^2 = c^2(1 + dx^2y^2)$$

Long, long ago ...



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Elliptic Curves – p. 10

Euler 1761

“Observationes de Comparatione Arcuum Curvarum Irrectificabilium”

I. DE ELLIPSI

1. Sit quadrans ellipticus ABC (Fig. 1), cuius centrum in C , eiusque semiaxes ponantur $CA=1$ et $CB=c$; sumta ergo abscissa quacunq̄ue $CP=x$ erit applicata ei respondens $PM=y=c\sqrt{1-xx}$; cuius differentiale cum sit $dy = -\frac{cx dx}{\sqrt{1-xx}}$, erit abscissae $CP=x$ arcus ellipticus respondens

$$BM = \int \frac{dx \sqrt{1-(1-cc)xx}}{\sqrt{1-xx}}$$

Ponatur brevitatis gratia $1-cc=n$, ut sit arcus

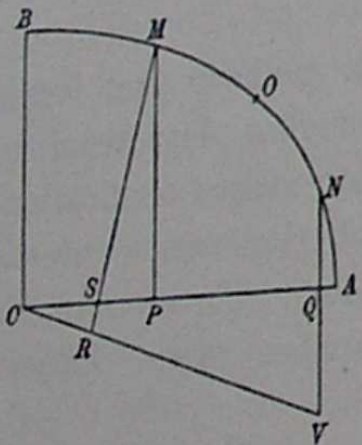
$$BM = \int dx \sqrt{\frac{1-nxx}{1-xx}}$$


Fig. 1.

$$\frac{1}{y^2} = \frac{1-nx^2}{1-x^2} \Leftrightarrow x^2 + y^2 = 1 + nx^2y^2.$$

Euler 1761

COROLLARIUM 3

43. Inventio ergo cordarum arcuum quorumvis multiplo- rum una cum cordis complementi ita se habebit:

Corda arcus	Corda complementi
simplici = a	simplici = A
dupli = $b = \frac{2aA}{1 - aaAA}$	dupli = $\frac{AA - aa}{1 + aaAA} = B$
triplici = $c = \frac{aB + bA}{1 - abAB}$	triplici = $\frac{AB - ab}{1 + abAB} = C$
quadrupli = $d = \frac{aC + cA}{1 - acAC}$	quadrupli = $\frac{AC - ac}{1 + acAC} = D$
quintupli = $e = \frac{aD + dA}{1 - adAD}$	quintupli = $\frac{AD - ad}{1 + adAD} = E$
etc.	etc.

Euler gives doubling and (special) addition for (a, A) on $a^2 + A^2 = 1 - a^2A^2$.

Gauss, posthumously

ELEGANTIORES INTEGRALIS $\int \frac{dx}{\sqrt{(1-x^4)}}$ PROPRIETATES.



[2.]

$$1 = ss + cc + ssc c \quad \text{sive} \quad 2 = (1 + ss)(1 + cc) = \left(\frac{1}{ss} - 1\right)\left(\frac{1}{cc} - 1\right)$$

$$s = \sqrt{\frac{1-cc}{1+cc}}, \quad c = \sqrt{\frac{1-ss}{1+ss}}$$

$$\sin \operatorname{lemn}(a \pm b) = \frac{sc' \pm s'c}{1 \mp scs'c'}$$

$$\cos \operatorname{lemn}(a \pm b) = \frac{cc' \mp ss'}{1 \pm s's'cc'}$$

$$\sin \operatorname{lemn}(-a) = -\sin \operatorname{lemn} a, \quad \cos \operatorname{lemn}(-a) = \cos \operatorname{lemn} a$$

$$\sin \operatorname{lemn} k\omega = 0 \quad \sin \operatorname{lemn}(k + \frac{1}{2})\omega = \pm 1$$

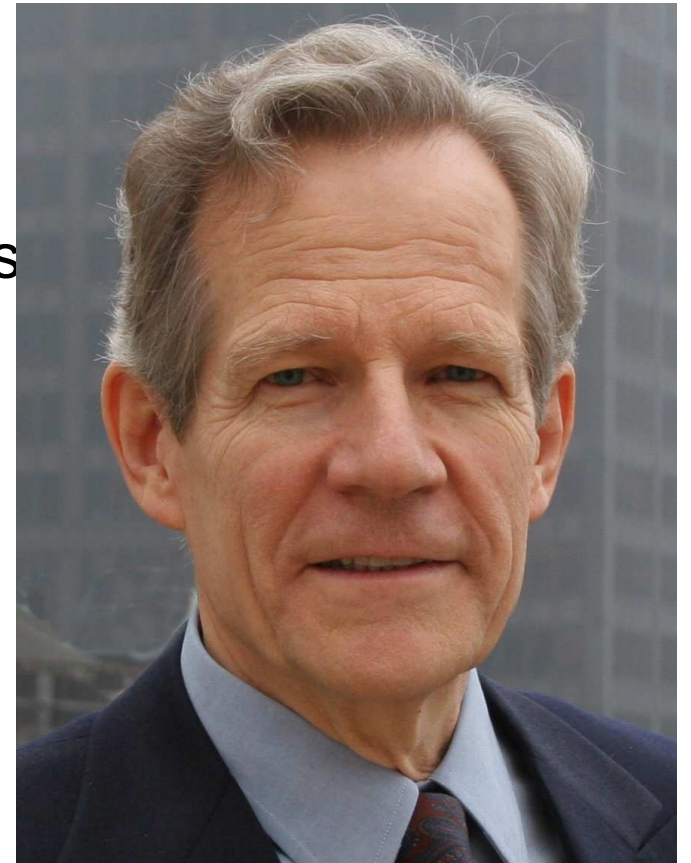
$$\cos \operatorname{lemn} k\omega = \pm 1 \quad \cos \operatorname{lemn}(k + \frac{1}{2})\omega = 0$$

Gauss gives general addition for arbitrary points on

$$1 = s^2 + c^2 + s^2c^2.$$

Harold M. Edwards

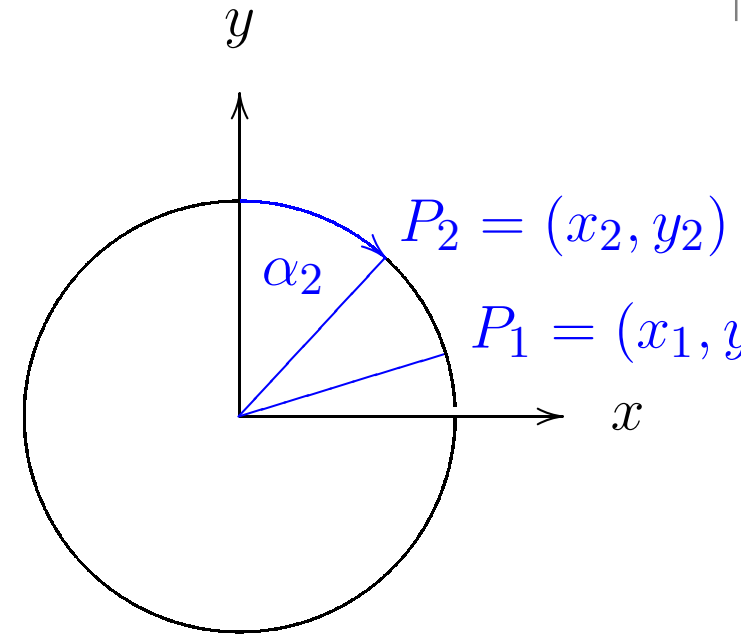
- The Gauss/Euler example is mentioned in some books.
- Edwards generalized this single example to whole class of curves
- Showed how to do arithmetic on this curve.
- Shows that – after some field extensions – every elliptic curve over field k of odd characteristic is birationally equivalent to a curve of the form
$$x^2 + y^2 = a^2(1 + x^2y^2), a^5 \neq a$$
- Many more properties in his paper
Bulletin of the AMS, **44**, 393–422, 2007



Do you know how to add on a circle?

Let k be a field with $2 \neq 0$.

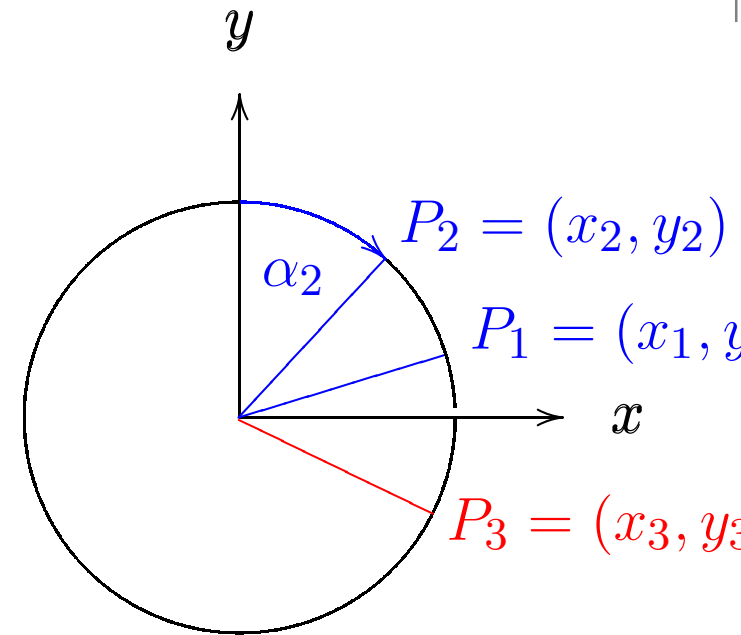
Circle: $\{(x, y) \in k \times k \mid x^2 + y^2 = 1\}$



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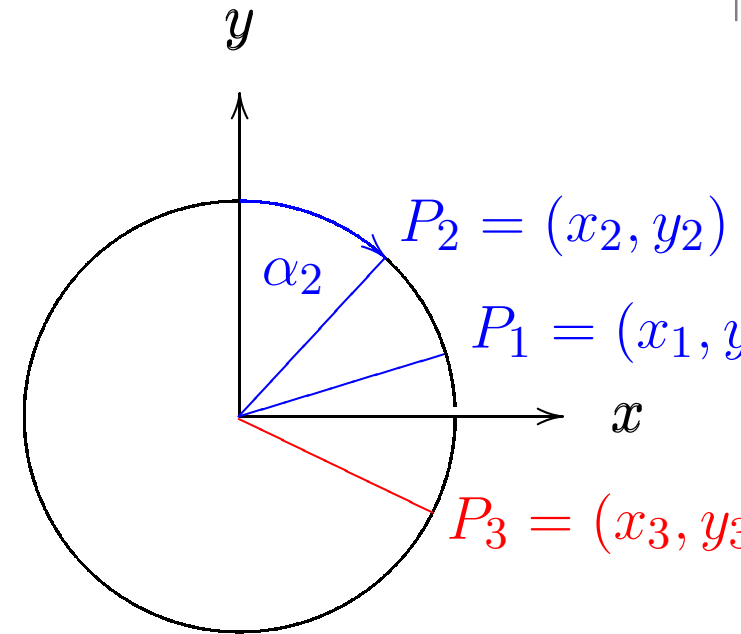
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$x_i = \sin(\alpha_i)$, $y_i = \cos(\alpha_i)$

$$\begin{aligned}x_3 &= \sin(\alpha_1 + \alpha_2) \\ &= \sin(\alpha_1) \cos(\alpha_2) + \cos(\alpha_1) \sin(\alpha_2)\end{aligned}$$

$$\begin{aligned}y_3 &= \cos(\alpha_1 + \alpha_2) \\ &= \cos(\alpha_1) \cos(\alpha_2) - \sin(\alpha_1) \sin(\alpha_2)\end{aligned}$$



Addition of angles defines commutative group law

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$, where

$$x_3 = x_1 y_2 + y_1 x_2 \text{ and } y_3 = y_1 y_2 - x_1 x_2.$$

Now add on an Edwards curve

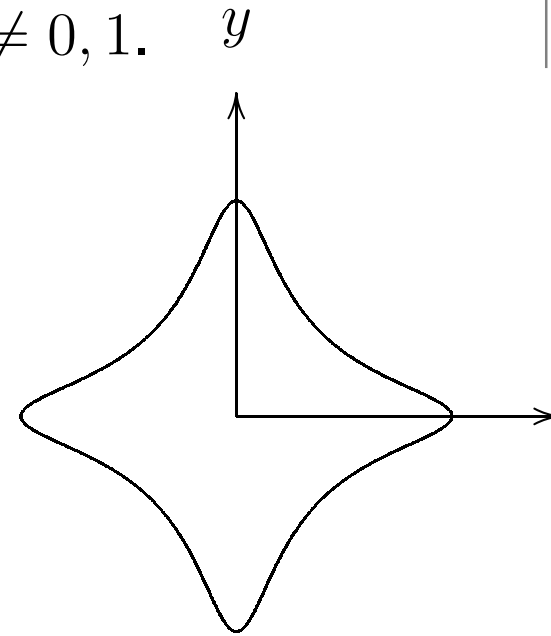
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Edwards curve:

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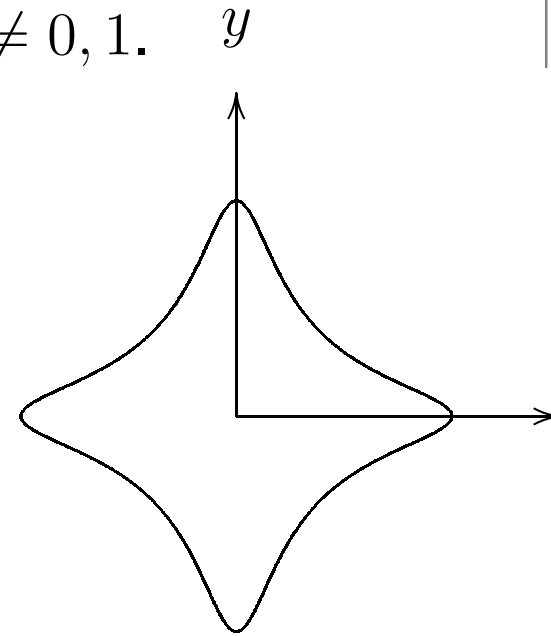
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Associative operation on points

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$$

defined by **Edwards addition law**

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2} \text{ and } y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$



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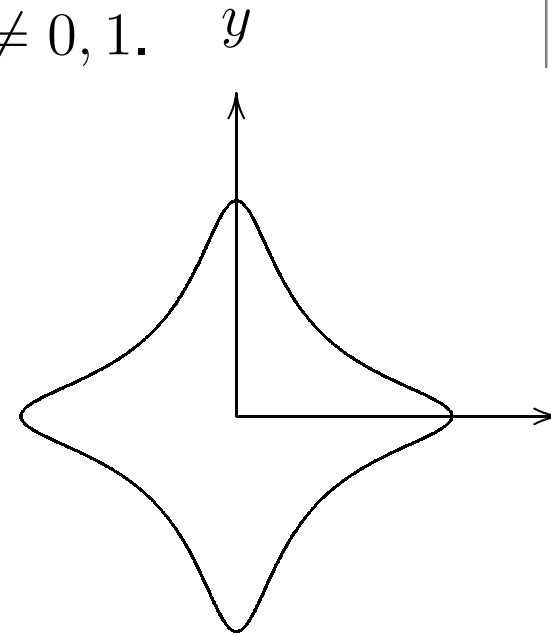
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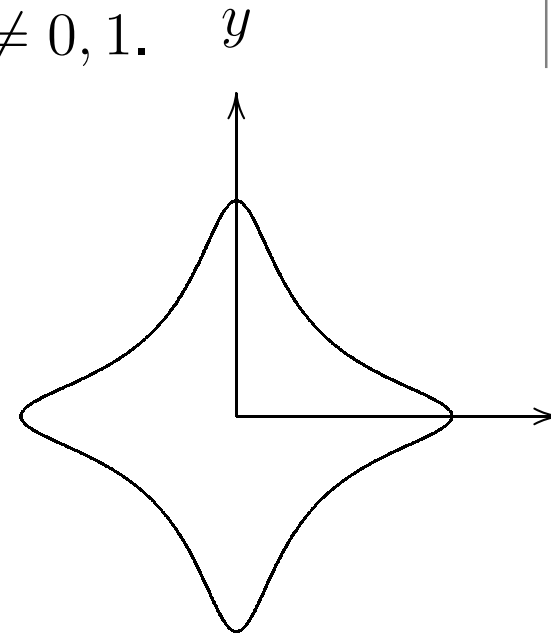
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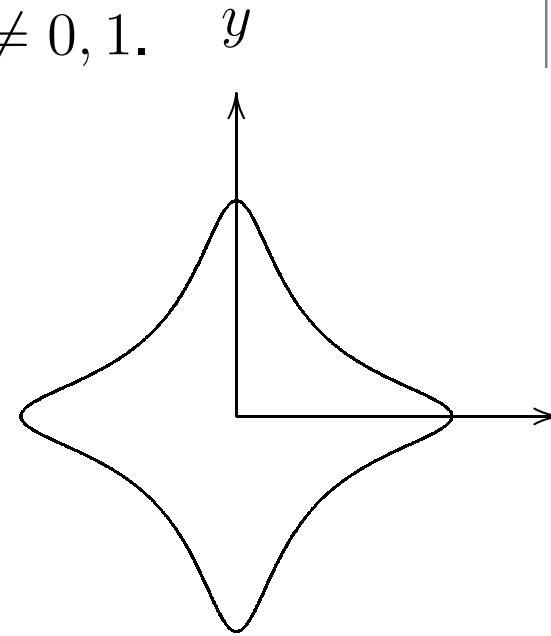
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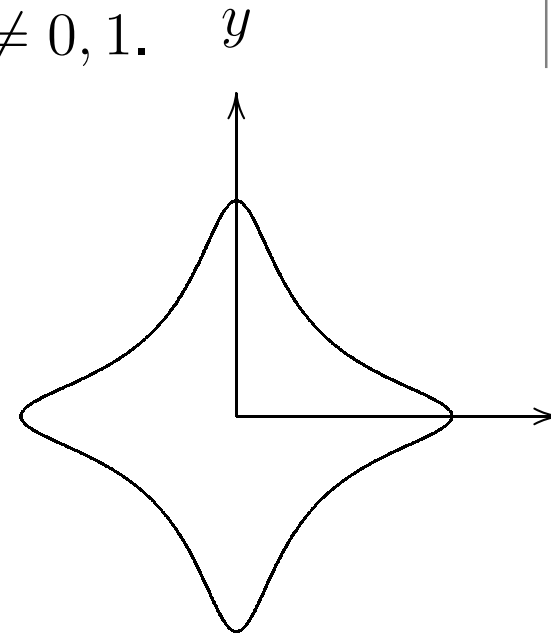
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- Neutral element is $(0, 1)$ (like on circle).
- $-(x_1, y_1) = (-x_1, y_1)$ (like on circle).



Relationship to elliptic curves

- Every elliptic curve with point of order 4 is birationally equivalent to an Edwards curve.
- Let $P_4 = (u_4, v_4)$ have order 4 and shift u s.t. $2P_4 = (0, 0)$. Then Weierstrass form:

$$v^2 = u^3 + (v_4^2/u_4^2 - 2u_4)u^2 + u_4^2u.$$

- Define $d = 1 - (4u_4^3/v_4^2)$.
- The coordinates $x = v_4u/(u_4v)$, $y = (u - u_4)/(u + u_4)$ satisfy

$$x^2 + y^2 = 1 + dx^2y^2.$$

- Inverse map $u = u_4(1 + y)/(1 - y)$, $v = v_4u/(u_4x)$.
- Finitely many exceptional points. Exceptional points have $v(u + u_4) = 0$.

Nice features of the addition law

- Neutral element of addition law is affine point, this avoids special routines (for $(0, 1)$ one of the inputs or the result).
- Addition law is symmetric in both inputs.
- $P + Q = \left(\frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2}, \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2} \right)$.

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- No reason that the denominators should be 0.
- Addition law produces correct result also for doubling.
- **Unified group operations!**
- Having addition law work for doubling removes some checks from the code.

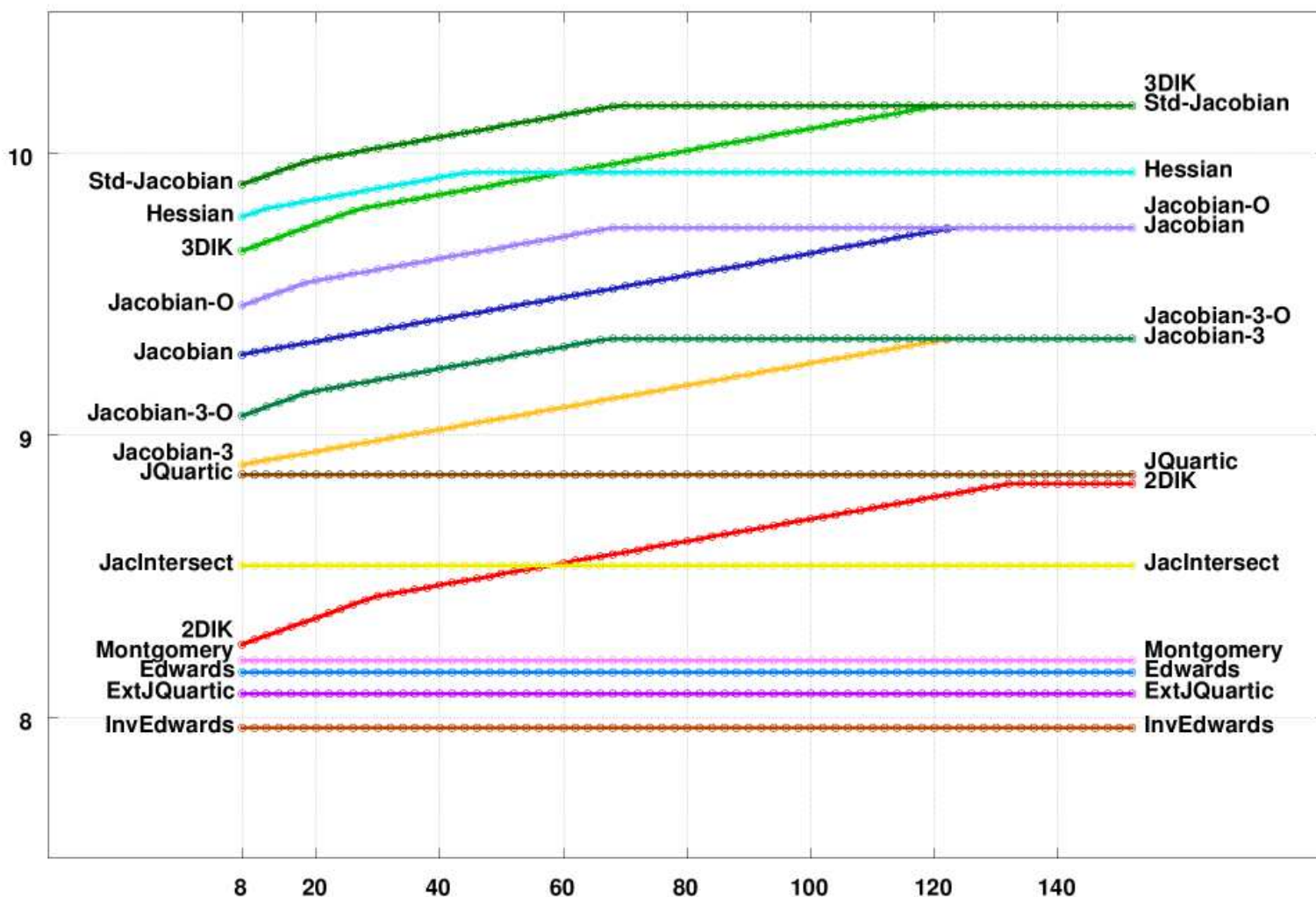
Complete addition law

- For d not a square in k , the Edwards addition law is **complete**, there are no exceptional cases.
- Edwards addition law allows omitting all checks
 - Neutral element is affine point on curve.
 - Addition works to add P and P .
 - Addition works to add P and $-P$.
 - Addition just works to add P and any Q .
- Only complete addition law in the literature.
- Very fast point addition $10M + 1S + 1D$.
- Dedicated doubling formulas need only $3M + 4S$.
- Fastest scalar multiplication in the literature, in particular in 'Inverted Edwards' representation.

Speed comparison & influence of inversion

Joint work with D.J. Bernstein

Field multiplications per bit (single scalar, 256 bits) as function of I/M, assuming S/M = 0.8



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Elliptic Curves – p. 20

Consequences

- Speed records for Diffie-Hellman key exchange.
- Faster signature computation (additions in this form are easy).
- Can afford larger key length for same speed (or get faster for same level of security).
- Applications using scalar multiplication on elliptic curves get faster ...

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- Applications using scalar multiplication on elliptic curves get faster ...
- ... which brings us to the second topic of this talk

RSA Cryptosystem

Secret computation – key generation:

- pick two primes p, q ,
- compute $N = p \cdot q$,
- compute $\varphi(N) = (p - 1)(q - 1)$, where φ is the Euler- φ function,
- pick integer e co-prime to $\varphi(N)$,
- compute d with $de \equiv 1 \pmod{\varphi(N)}$.

Note that for any m with $\gcd(m, N) = 1$ the order of m modulo N is $\varphi(N)$.

So we have

$$(m^e)^d \equiv m^{ed} \equiv m^{k\varphi(N)+1} \equiv m \pmod{N},$$

for some integer k .

Pitfalls ctd.

- Yes, this looks like very close to a power of 10, actually close to 10^{340} . Squareroot \sqrt{N} is almost an integer, almost 10^{170} .

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- Yes, this looks like very close to a power of 10, actually close to 10^{340} . Squareroot \sqrt{N} is almost an integer, almost 10^{170} .
- Brute-force search $\gcd(N, 10^{170} - i)$ finds factor $q = 10^{170} - 33$ and then $p = N/q = 10^{170} + 63$.
- This attack is well known and mentioned e.g. in the Handbook of Applied Cryptography. Yet it is found in practice by amateurs, usually with powers of 2 rather than 10.
- Famous example: letter bomber Franz Fuchs in Austria (6 attack series between 1993 and capture in 1997) send RSA-encrypted letter to Austrian magazine 'Profil'. Modulus N had 1024 bits, i.e. full size, but both primes were extremely close to 2^{512} . Found by Hans Dobbertin.

Attacks against generic RSA

Best known attacks against generic RSA try to find a, b with

$$a^2 \equiv b^2 \pmod{N}, \text{ i.e., } N \mid (a^2 - b^2). \quad (1)$$

If $N \nmid (a \pm b)$ then

$$1 < \gcd(N, a - b) < N$$

finds nontrivial factor.

The quadratic sieve finds such a relation by computing for many integers a_i

$$(a_i^2 \pmod{N}) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} a'_i, \quad (2)$$

where p_j in factor base F and $\gcd(a'_i, p_j) = 1$ for all $p_j \in F$.

Collect relations of form (2) with $a'_i = 1$; construct relation (1) using linear algebra. Number field sieve is similar.

Computational issues in factorization

- The factorization $N = p \cdot q$ is broken down into many factorizations of the form

$$(a_i^2 \bmod N) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} a_i'$$

- Find small factors $p_i < 2^{20}$ by trial division/sieving.
- Find larger factors $p_i < 2^{40}$ by using ECM - the Elliptic Curve Method of factorization.
- Main idea of ECM: to factor $b = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ compute $[k]P$ for some point P on an elliptic curve E modulo b ; hope that k is a multiple of the order of P on E modulo p_i and not on E modulo p_j for some i and j .
- Retrieve p_i by a gcd computation (technically all factors p_i such that k is a multiple of the group order of E modulo p_i .)

Impact of faster arithmetic on EC

- Traditionally the computation $[k]P$ was carried out in Montgomery form elliptic curves; Edwards curves are solidly faster.
- Previously fastest implementation was GMP-ECM by Zimmerman et al.
- Using Edwards curves and some new ideas we (joint work with D.J. Bernstein, P. Birkner, C. Peters) sped up the best known ECM implementation; faster arithmetic alone gives 7%; we also found more effective curves.
- Latest results: our cluster is warming up with some EECM (Edwards-ECM) computations (stage 1 and 2 working) searching for 40-bit factors.

Edwards curves

—
blessing to ECC, harm to RSA

References

- Bernstein/L. “Faster addition and doubling on elliptic curves”, Asiacrypt 2007, pp. 29-50. Online.
- Bernstein/Birkner/L./Peters, “Optimizing double-base elliptic-curve single-scalar multiplication”, Indocrypt 2007, pp. 167–182. Online.
- Bernstein/L. “Analysis and optimization of elliptic-curve single-scalar multiplication”, Proceedings of Fq8.
- Bernstein/Birkner/L./Peters, “Twisted Edwards Curves”, Africacrypt.
- Bernstein/Birkner/L./Peters, “ECM using Edwards curves”.
- Bernstein/L./Rezaeian Farashahi, “Binary Edwards Curves”.

Details on the blow-up

• Points with $v(u + u_4) = 0$ on Weierstrass curve map to points at infinity on desingularization of Edwards curve.

• Reminder: $d = 1 - (4u_4^3/v_4^2)$.

• $u = -u_4$ is u -coordinate of a point iff

$$\begin{aligned} & (-u_4)^3 + (v_4^2/u_4^2 - 2u_4)(u_4)^2 + u_4^2(u_5) \\ &= v_4^2 - 4u_4^3 = v_4^2 d \end{aligned}$$

is a square, i. e., iff d is a square.

• $v = 0$ corresponds to $(0, 0)$ which maps to $(0, -1)$ on Edwards curve and to solutions of $u^2 + (v_4^2/u_4^2 - 2u_4)u + u_4^2 = 0$. Discriminant is

$$(v_4^2/u_4^2 - 2u_4)^2 - 4u_4^2 = v_4^4 d,$$

i. e., points defined over k iff d is a square.

Back to Edwards curves

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + dx_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - dx_1 x_2 y_1 y_2} \right)$$

What if denominators are 0?

Back to Edwards curves

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What if denominators are 0?

Answer: They are never 0 if d is not a square in k .

Back to Edwards curves

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + dx_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - dx_1 x_2 y_1 y_2} \right)$$

What if denominators are 0?

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Intuitive explanation:

The points at infinity (for which the addition law must fail) have minimal field of definition $k(\sqrt{d})$.

Back to Edwards curves

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Explicit proof: Let $(x_1, y_1), (x_2, y_2)$ be on curve, i.e., if $x_i^2 + y_i^2 = 1 + dx_i^2 y_i^2$. Write $\epsilon = dx_1 x_2 y_1 y_2$ and suppose $\epsilon \in \{-1, 1\}$. Then $x_1, x_2, y_1, y_2 \neq 0$ and $dx_1^2 y_1^2 (x_2^2 + y_2^2) = dx_1^2 y_1^2 + d^2 x_1^2 y_1^2 x_2^2 y_2^2$

Back to Edwards curves

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$$\begin{aligned} dx_1^2 y_1^2 (x_2^2 + y_2^2) &= dx_1^2 y_1^2 + d^2 x_1^2 y_1^2 x_2^2 y_2^2 \\ &= dx_1^2 y_1^2 + \epsilon^2 \end{aligned}$$

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Back to Edwards curves

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Back to Edwards curves

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$dx_1^2 y_1^2 (x_2^2 + y_2^2) = x_1^2 + y_1^2$, so

$$\begin{aligned} (x_1 + \epsilon y_1)^2 &= x_1^2 + y_1^2 + 2\epsilon x_1 y_1 = dx_1^2 y_1^2 (x_2^2 + y_2^2) + 2x_1 y_1 dx_1 x_2 y_1 y_2 \\ &= dx_1^2 y_1^2 (x_2^2 + 2x_2 y_2 + y_2^2) = dx_1^2 y_1^2 (x_2 + y_2)^2. \end{aligned}$$

Back to Edwards curves

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$$x_2 + y_2 \neq 0 \Rightarrow d = ((x_1 + \epsilon y_1)/x_1 y_1 (x_2 + y_2))^2 \Rightarrow d = \square$$

$$x_2 - y_2 \neq 0 \Rightarrow d = ((x_1 - \epsilon y_1)/x_1 y_1 (x_2 - y_2))^2 \Rightarrow d = \square$$

If $x_2 + y_2 = 0$ and $x_2 - y_2 = 0$ then $x_2 = y_2 = 0$, contradiction.