Semiring Neighbours

An Algebraic Embedding and Extension of Neighbourhood Logic

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Aim

- embed and extend NL
- get additional results for NL
- generalise the existing results for NL
- adopt the results to other areas, like graphs and hybrid systems
1 About Neighbourhood Logic

- purpose: reasoning about time intervals
- in particular, about neighbouring intervals
- chop-based interval temporal logics, like ITL and IL, cannot express all desired properties
- first-order interval logic
- introduced by Zhou and Hansen in 1996
  expanded by Zhou and Roy
- main idea: left and right neighbourhoods as primitive intervals
left neighbourhood: $\lozenge_1 \phi$

$\lozenge_1 \phi$ holds on $[b, e]$ iff there exists $a \leq b$ such that $\phi$ holds on $[a, b]$

right neighbourhood: $\lozenge_r \phi$

$\lozenge_r \phi$ holds on $[b, e]$ iff there exists $c \geq e$ such that $\phi$ holds on $[e, c]$
- expanding modalities
- neighbours only depends on contact points
- characterise those as action points of sequential composition
2 Some short definitions

Definition 2.1 idempotent semiring \((S, +, \cdot, 0, 1)\):

- \((S, +, 0)\) commutative monoid \((+ \text{ is choice})\)
- \((S, \cdot, 1)\) monoid \((\cdot \text{ is composition})\)
- multiplication is distributive
  \[(a + b) \cdot c = a \cdot c + b \cdot c \quad a \cdot (b + c) = a \cdot b + a \cdot c\]
- \(0\) is annihilator
  \[0 \cdot a = 0 = 0 \cdot a\]
- \(a + a = a\)

natural order: \(a \leq b \iff a + b = b\)
Definition 2.2 *test semiring* \((S, \text{test}(S))\):

- \(S\) idempotent semiring
- \(\text{test}(S) \subseteq [0, 1]\) Boolean algebra (abstract assertions)

Definition 2.3 *domain semiring* \((S, \lceil \rceil)\):

- \(S\) test semiring
- \(\lceil : S \rightarrow \text{test}(S)\)

\[ a \leq \lceil a \cdot a \rceil \quad \lceil p \cdot a \rceil \leq p \]

analogously: *codomain semiring*

work on semirings with domain *and* codomain

e.g.: [DesharnaisMöllerStruth04]
2.1 Examples

- **algebra of intervals**
  - elements: sets of intervals
  - \( \uparrow \): starting points, i.e., \( \uparrow I = \{ [a, a] : [a, x] \in I \} \)
  - \( \downarrow \): ending points

- **algebra of binary relations under relational composition**
  - \( \uparrow \): domain of a relation, i.e., \( \uparrow R = \{(a, a) : (a, x) \in R \} \)
  - \( \downarrow \): range of a relation

- **path algebra under path fusion**
  - elements: sets of paths in a given graph
  - \( \uparrow \): starting nodes
  - \( \downarrow \): ending points

- ...
3  Embedding of NL

- b is starting point of [b, e]
- b is ending point of [a, b]

Definition 3.1

x is a left neighbour of y (or for short: \( x \leq \Diamond_1 y \)) iff \( x \preceq \llbracket y \rrbracket \)

x is a right neighbour of y (or for short: \( x \leq \Diamond_r y \)) iff \( \llbracket x \rrbracket \leq y \)
Let $[\phi]$ be the set of all intervals where $\phi$ holds.

**Lemma 3.2**

$$\Diamond_r \phi \text{ holds on } x \iff x \leq \Diamond_1 [\phi] \iff x \uparrow \leq \lceil([\phi]) \rceil$$

$$\Diamond_1 \phi \text{ holds on } x \iff x \leq \Diamond_r [\phi] \iff x \downarrow \leq ([\phi]) \lceil$$

- thus, NL is embedded into semirings
- NL can be adopted to other interpretations, like graphs
more neighbours (briefly)

- perfect neighbours (box operators)
  \[
  \Box_1 \phi \overset{\text{def}}{=} \neg \Diamond_1 \neg \phi \text{ holds on } x \iff (\neg \phi) \cdot \neg x \leq 0
  \]
  \[
  \Box_r \phi \overset{\text{def}}{=} \neg \Diamond_r \neg \phi \text{ holds on } x \iff x \cdot (\neg (\neg \phi)) \leq 0
  \]
  \[
  x \leq \Box_1 y \iff \neg y \cdot \neg x \leq 0
  \]
  \[
  x \leq \Box_r y \iff x \cdot \neg y \leq 0
  \]

- combinations \( \Diamond_1 \Diamond_r \phi, \Diamond_r \Diamond_1 \phi \)
  \[
  \neg x \leq \neg y\quad x \leq y
  \]

- boxes of combinations
  \[
  \neg x \cdot \neg y \leq 0\quad \neg y \cdot x \leq 0
  \]
4 Results

- underlying theory
  - de Morgan dualities
  - Galois connections, like
    \[ \Diamond_l x \leq y \iff x \leq \Box_r y \quad \Diamond_r x \leq y \iff x \leq \Box_l y \]

- simplifying NL
  - at least two axioms of NL can be dropped
  - additional box operators
  - most of the properties of [ZhouHansen] follow from the Galois connections
  - there are explicit expressions for neighbours,
    e.g. \( \Diamond_l y = \top \cdot \bar{y} \)

- almost all results of NL can be lifted to semirings
5 Interpretation in other semirings

- algebra of binary relations:
  - neighbours: permeability
    \[ \Diamond^r R = \{(x, y) \mid \exists w : (w, x) \in R, y \in M \} \]
  - perfect neighbours: full permeability
    \[ \Box^r R = \{(x, y) \mid \forall w : (w, x) \in R, y \in M \} \]
  - the relation you can only reach through \( R \)

- path algebra:
  - neighbours: reachability
  - perfect neighbours: exclusive reachability
Interpretation in hybrid systems

- a *trajectory* is a pair \((i, f)\) of an interval \(i\) and a function 
  \(f : i \rightarrow V\)
- only finite trajectories!
- \((\mathcal{P}(\text{TRA}), \cup, \circ, \emptyset, \mathbb{1})\) forms a semiring
- allows statements about neighbours
  (similar to the original idea)
- interpretation of neighbours:
  again some kind of reachability

[HöfnerMöller05]
6 Conclusion and Outlook

done

- embed NL into semirings
- expanded NL by additional operators and theorems
- simplified NL (some axioms are theorems in our approach)

to do

- expand neighbours to Lazy semirings [Möller04]
- include infinite elements
- get interpretation of hybrid systems with trajectories of finite and infinity length [HöfnerMöller05]