A Linear Bound on the Diameter of the Transportation Polytope

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ABSTRACT:
The transportation problem (TP) is a classic problem in operations research. The problem was posed for the first time by Hitchcock in 1941 [Hitchcock, 1941] and independently by Koopmans in 1947 [Koopmans, 1948], and appears in any standard introductory course on operations research.

The $m \times n$ TP has m supply points and n demand points. The objective is to minimize costs of transportation from supply points to demand points, given unit transportation cost between each such pair, such that all supply and demand restrictions are satisfied, assuming equality of total supply and demand. The set of feasible solutions of TP, is called the transportation polytope.

The diameter of the polytope is the combinatorial diameter of the graph with vertices the vertices of the polytope and edges its 1-dimensional faces. In 1957 W.M. Hirsch stated his famous conjecture (cf. [Dantzig, 1963]) that any d-dimensional polytope with n facets has diameter at most $n - d$. So far the best bound for any polytope is $O(n^{\log d+1})$ [Kalai and Kleitman, 1992]. Any strongly polynomial bound is still lacking. Such bounds exist for special classes of polytopes, among which special classes of transportation polytopes [Balinski, 1974],[Bolker, 1972] and the polytope of the dual of TP [Balinski, 1974].

Strongly polynomial bounds on the diameter of the transportation polytope are $O(m^{16}\, n^3\, (\ln(mn))^3)$ [Dyer and Frieze, 1994] and $O(nm(n+m)\log(n+m))$ [Orlin, 1997].

We will give a very simple proof of a quadratic bound on the diameter of the transportation polytope. Then we briefly sketch the first linear bound of $8(m + n - 1)$ by Brightwell, van den Heuvel and S. Recently, Hurkens refined our analysis and diminished the bound by a factor of 2, arriving at $4(m+n-2)$. We will present his proof in detail. All proofs given are constructive: they give algorithms that describe how to go from any vertex to any other vertex on the transportation polytope in less than the claimed number of steps along the edges. According to the Hirsch Conjecture the bound on the TP polytope should be $m + n - 1$. Thus, we are within a factor 4 of the Hirsch bound.