Dynamic Capillarity in Porous Media

Description of the research

In my thesis, I have addressed mathematical and numerical analysis questions related to non-standard porous media flow models, and investigated the effect of different capillary pressure assumptions. Based on mass conservation and Darcy's law, we obtained the following non-dimensional partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(H(u) \frac{\partial p_c}{\partial x} \right). \tag{1}$$

Here u stands for saturation of one phase, f and H are the flow function, respectively the diffusion function, both being nonnegative and bounded. One important issue here is H could be 0. P_c denotes the capillary pressure and it is considered as a function of u in conventional models. In our research, we focus on the model suggested by S.M. Hassanizadeh and W.G. Gray, where $p_c = p_c^{static} + p_c^{dynamic}$ and particularly we investigate the case

$$p_c^{static} = u \text{ and } p_c^{dynamic} = \varepsilon \tau u_t. \tag{2}$$

Observe that equation (1) can be considered as a regularization of the hyperbolic Buckley-Leverett (BL) equation. In my thesis, we focus on the following issues:

1. Travelling wave (TW) solutions. In the case $H > 0$, we investigate the relationship between τ, and the states u_l, u_r, for which standard TW solutions exist. We continue with the degenerate case, where H may become 0 for certain values of u, and obtain similar results. Furthermore, we show that for large enough values of τ, standard TW solutions cease to exist. Then sharp TW solutions are introduced and the existence is shown. Numerical simulations for both classical and sharp TW solutions are provided.

2. One intermediate model, where the degeneracy only appears in the second order term, is investigated. Specifically, the equation

$$u_t + \nabla \cdot \mathbf{F}(u) = \nabla \cdot (H(u)\nabla u) + \tau \Delta u_t. \tag{3}$$

is considered. The existence, uniqueness of weak solutions to (3), and the convergence of the corresponding numerical scheme are proved.

3. We prove the existence of weak solutions to the original equation (1). To overcome the difficulties due to degeneracy, a regularization method is employed. The existence for the regularized problem, as well as some a-priori estimates which are uniform w.r.t the regularization parameter, are obtained. Based on compensated compactness and equi-integrability arguments, the existence of the weak solutions is proved.

4. Numerical simulations using Euler-implicit method and Newton method have been done, and compared with the experimental results done by S. Bottero and S.M. Hassanizadeh. We further investigate different numerical schemes.