CONFORMAL RANDOM GEOMETRY
& QUANTUM GRAVITY

BERTRAND DUPLANTIER

Abstract.

The study of statistical mechanical systems at critical points in two dimensions leads to the consideration of conformally-invariant (CI) scaling curves, for example the boundaries of clusters. This set of lectures gives a comprehensive description of the fractal geometry of such CI scaling curves, in the plane or half-plane.

It focuses on deriving critical exponents associated with interacting random paths, by exploiting an underlying quantum gravity (QG) structure. It makes use of Knizhnik, Polyakov and Zamolodchikov (KPZ) maps relating exponents in the plane to those on a random lattice, i.e., in a fluctuating metric. This is done within the framework of conformal field theory (CFT), with applications to well-recognized critical models, like Brownian paths, $O(N)$ and Potts models, and to the Stochastic Löwner Evolution (SLE).

Two fundamental ingredients of the QG construction are the relation between bulk and Dirichlet boundary exponents, and establishing additivity rules for QG boundary conformal dimensions associated with mutually-avoiding random sets.

A general reference for the content of these lectures is [1].

The first lecture will be devoted to the non-intersection exponents for random walks (RW's) or Brownian paths, self-avoiding walks (SAW's), or arbitrary mixtures thereof in the plane. I shall give a description of the partition functions of collections of such walks on a random lattice, and calculate those partition functions by elementary applications of random matrix theory. From those and the KPZ relation I shall derive in particular the non-intersection exponents of Brownian paths in the plane [1, 2]. ¿ The general structure of the scaling behavior of the partition functions so derived will also be used to established general quantum gravity additivity rules for scaling exponents, to be used in the following lectures.

The second lecture will focus on the multifractal properties of the harmonic measure (i.e., electrostatic potential, or diffusion field) near any conformally invariant fractal boundary in the plane. The multifractal function $f(\alpha, c)$ gives the Hausdorff dimension of the set of points where the potential varies with distance r to the fractal frontier as r^{α}, and is given as a function of the central charge c of the associated CFT. It is obtained from the general QG approach described above.

Brownian paths, SAW's in the scaling limit, and critical percolation clusters all have identical spectra corresponding to the same central charge $c = 0$. The common Hausdorff dimension of their frontiers is $D = \sup_\alpha f(\alpha; c = 0) = 4/3$, which confirms Mandelbrot's conjecture for the Brownian frontier dimension. It has been proven rigorously by Lawler, Schramm, and Werner [3].

Higher multifractal functions, like the double spectrum $f_2(\alpha, \alpha'; c)$ of the double-sided harmonic measure, will also be considered.

The third lecture will deal with the universal mixed multifractal spectrum $f(\alpha, \lambda; c)$ describing the local winding rate λ and singularity exponent α of the harmonic measure near any CI scaling curve [4]. It gives a probabilistic description of the geometry of equipotentials near the CI curve, which appear as a collection of logarithmic spirals of varying rates λ.

The Hausdorff dimensions D_{H} of a non-simple scaling curve or cluster hull, and D_{EP} of its external perimeter or frontier, obey the duality equation $(D_{\mathrm{H}} - 1)(D_{\mathrm{EP}} - 1) = \frac{1}{4}$, valid for any value of the central charge c.

The duality which exists between simple and non-simple random paths is established via an extended KPZ relation for the SLE. It reflects a duality property $\kappa \to \kappa' = 16/\kappa$ for the SLE_κ, where the $\mathrm{SLE}_{\kappa'<4}$ is the frontier of the non-simple $\mathrm{SLE}_{\kappa>4}$ path. This allows one to calculate the SLE multifractal exponents from simple QG rules.

Finally, I address the question of the mathematically rigorous derivation of the multifractal spectra for the SLE.

References

[1] B. Duplantier, *Conformal fractal geometry and boundary quantum gravity*, in "Fractal geometry and Applications: A Jubilee of Benoît Mandelbrot", Proceedings of Symposia in Pure Mathematics, vol. 72, Part 2, M. L. Lapidus and M. van Frankenhuijsen eds., AMS, 2004; also at arXiv:math-phys/0303034.

[2] G. F. Lawler, O. Schramm, and W. Werner, Acta Math. **187**, (I) 237-273, (II) 275-308 (2001), arXiv:math.PR/9911084, arXiv:math.PR/0003156; Ann. Inst. Henri Poincaré PR **38**, 109-123 (2002), arXiv:math.PR/0005294.

[3] G. F. Lawler, O. Schramm, and W. Werner, Math. Res. Lett. **8**, 401-411 (2001), arXiv:math.PR/0010165; Acta Math. **189**, 179-201 (2002), arXiv:math.PR/0005295.

[4] B. Duplantier and I. A. Binder, Phys. Rev. Lett. **89**, 264101 (2002); arXiv:cond-mat/0208045.

Service de Physique Théorique, URA CNRS 2306, CEA/Saclay, F-91191 Gif-sur-Yvette Cedex, FRANCE, e-mail: bertrand@spht.saclay.cea.fr