CONFORMAL RANDOM GEOMETRY & QUANTUM GRAVITY

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MARK KAC SEMINAR

ON PROBABILITY AND PHYSICS

Utrecht

2004-2005

Mark Kac on *Probability and Physics* in:

Marian Smoluchowski and the Evolution
of Statistical Thought in Physics:

"··· in 1906 when Smoluchowski (influenced by the appearance of Einstein's two papers [on Brownian motion]) finally published his results, random phenomena would not come readily to mind. It required therefore, I think, an intellectual tour de force, to bring games of chance to bear upon understanding of physical phenomena."

RANDOM WALKS & QUANTUM GRAVITY

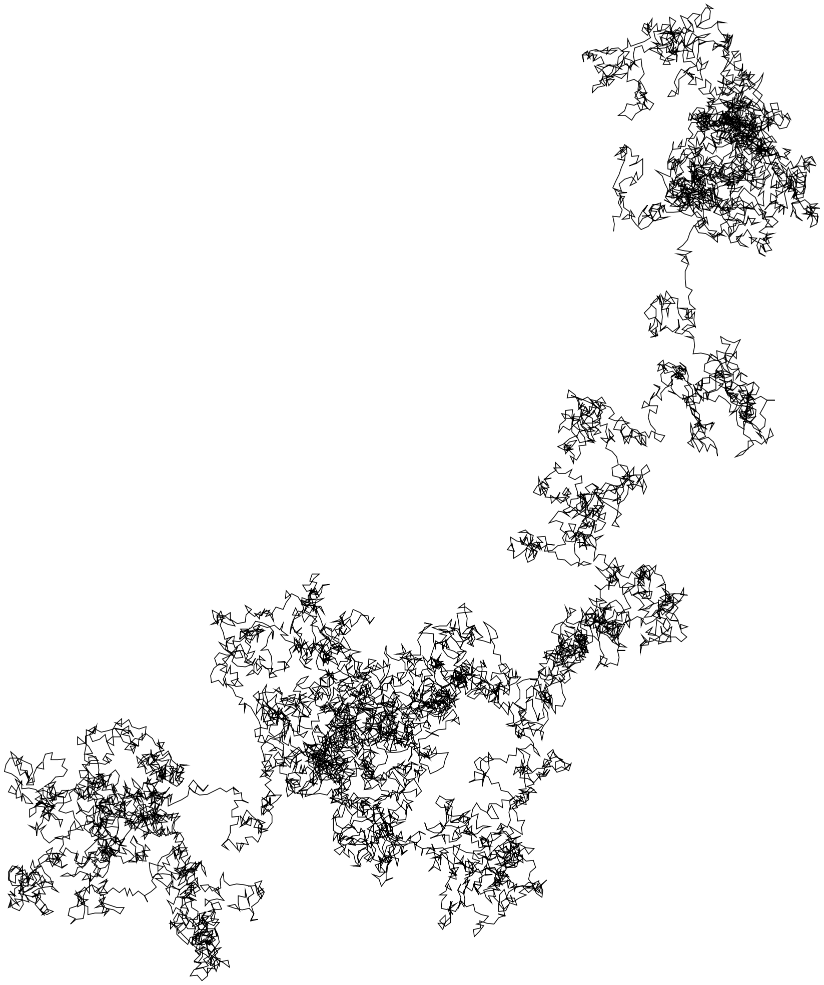
Mark Kac Seminars I & II

Utrecht

March 4, April 1st 2005

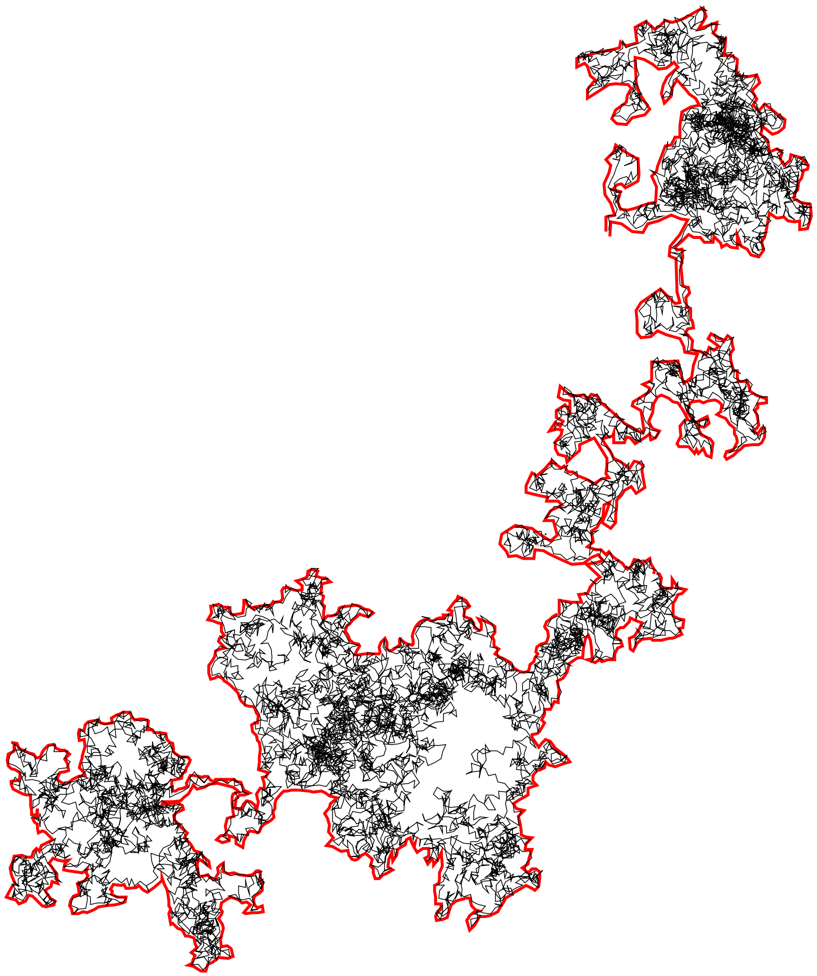
Random Walks

Brownian Path



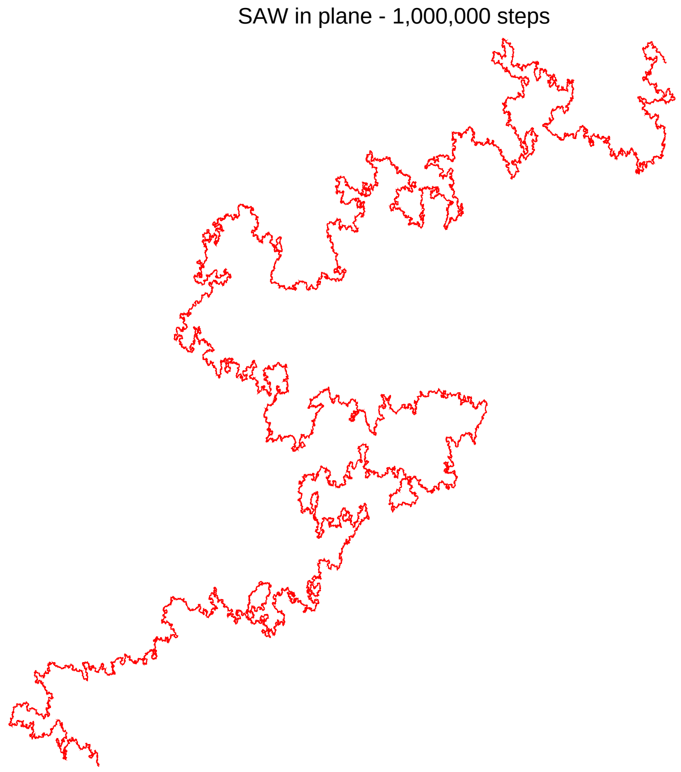
Paul Lévy: Conformal Invariance

Brownian Frontier



Mandelbrot conjecture (1982): Hausdorff dimension $D = \frac{4}{3}$, as a SAW.

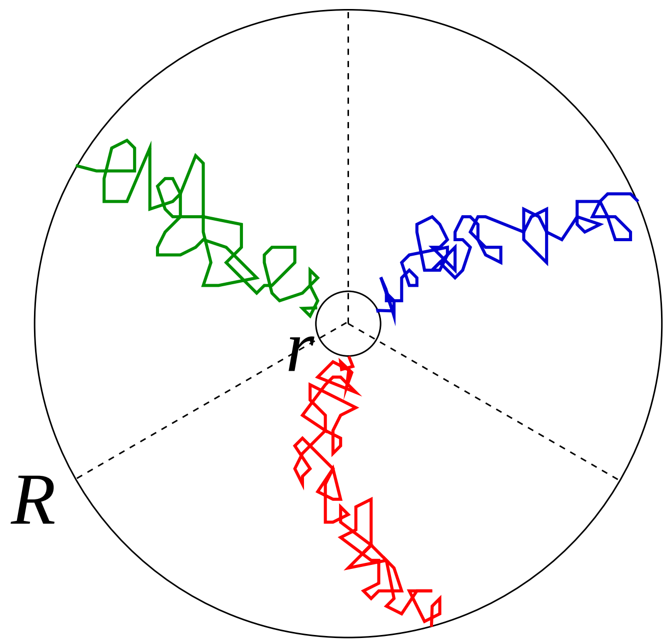
Self-Avoiding Walk



(courtesy of T. Kennedy)

B. Nienhuis (1982): $D = \frac{4}{3}$

Intersections of Random Walks



$L = 3$ non-intersecting random walks crossing an annulus from r to R

Probability

$$P_L(t) = P\left\{\cup_{l,l'=1}^{L}\left(B^{(l)}[0,t]\cap B^{(l')}[0,t]\right) = \emptyset\right\},$$

that the intersection of L paths $B^{(l)}$ is empty up to time t.

Scaling Exponents

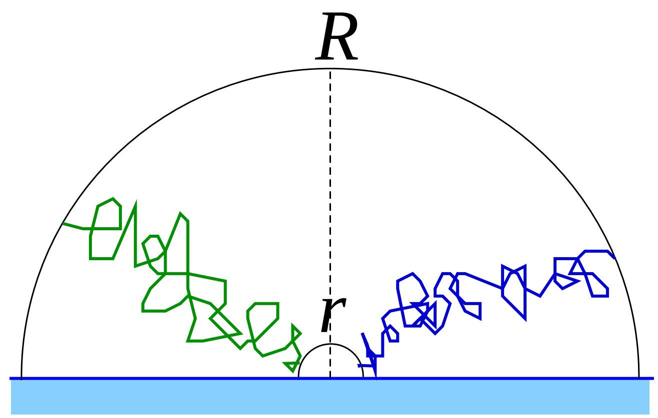
At large times, the non-intersection probability decays as

$$P_L(t) \approx t^{-\zeta_L},$$

where ζ_L is a *universal* exponent depending only on L. Similarly, the probability that the Brownian paths altogether traverse the annulus $\mathbb{D}(r,R)$ in $\mathbb{C}$ from the inner boundary circle of radius r to the outer one at distance R scales as

$$P_L(R) \approx (r/R)^{2\zeta_L}.$$

Half-Plane Case



$L = 2$ mutually-avoiding random walks crossing a half-annulus from r to R in the half-plane $\mathbb{H}$

L walks constrained to stay in the half-plane $\mathbb{H}$ with Dirichlet boundary conditions on $\partial\mathbb{H}$, and started at neighboring points near the boundary: non-intersection probability $\tilde{P}_L(t)$.

Boundary Exponents

Boundary critical exponent $\tilde{\zeta}_L$

$$\tilde{P}_L(t) \approx t^{-\frac{1}{2}\tilde{\zeta}_L}.$$

Probability that the Brownian paths altogether traverse the half-annulus $\tilde{\mathbb{D}}(r,R)$ in $\mathbb{H}$, centered on the boundary line ∂H, from the inner boundary circle of radius r to the outer one at distance R:

$$\tilde{P}_L(R) \approx (r/R)^{\tilde{\zeta}_L}.$$

Conformal Invariance and Weights

It was conjectured from conformal invariance arguments and numerical simulations that *(B. D.- Kwon (1988))*

$$\zeta_L = h_{0,L}^{(c=0)} = \frac{1}{24}\left(4L^2 - 1\right),$$

and for the half-plane

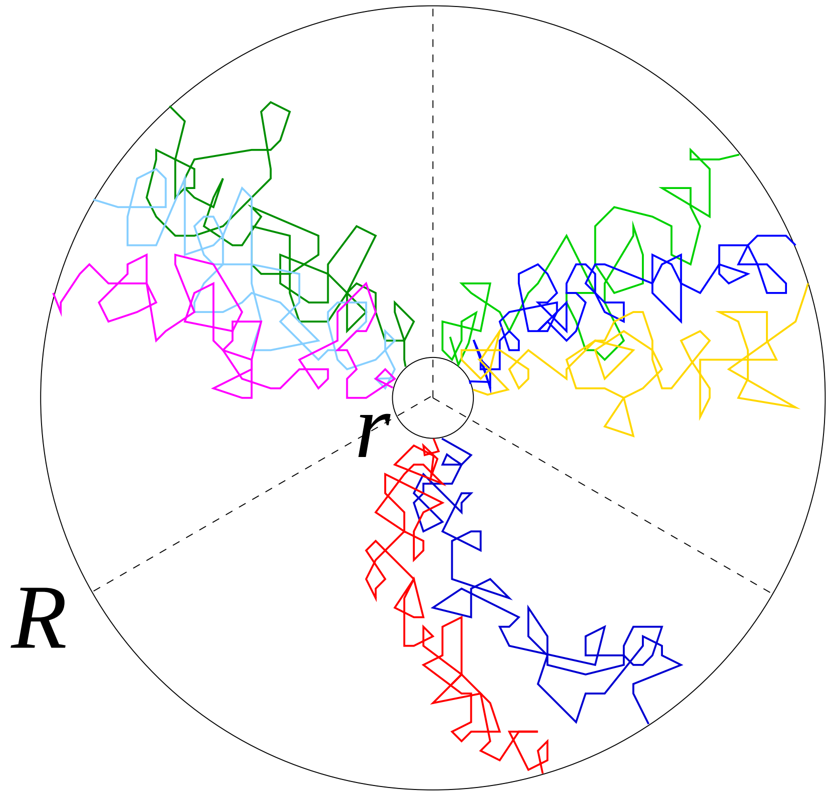
$$\tilde{\zeta}_L = h_{1,2L+2}^{(c=0)} = \frac{1}{3}L\left(1 + 2L\right),$$

where $h_{p,q}^{(c)}$ denotes the Kač conformal weight

$$h_{p,q}^{(c)} = \frac{\left[(m+1)p - mq\right]^2 - 1}{4m\left(m+1\right)},$$

of a minimal conformal field theory of central charge $c = 1 - 6/[m(m+1)]$, $m \in \mathbb{N}^*$. Brownian paths: $c = 0, m = 2$.

Non-Intersections of Packets of Walks



$L = 3$ packets of $n_1 = 3, n_2 = 3$, and $n_3 = 2$ independent planar random walks, in a mutually-avoiding star configuration, and crossing the annulus from r to R

Bulk Case

L mutually-avoiding packets $l = 1, \cdots, L$, made of n_l independent RW's, started at neighboring points.

Non-intersection probability of the L packets up to time t:

$$P_{n_1,\cdots,n_L}(t) \approx t^{-\zeta(n_1,\cdots,n_L)}$$

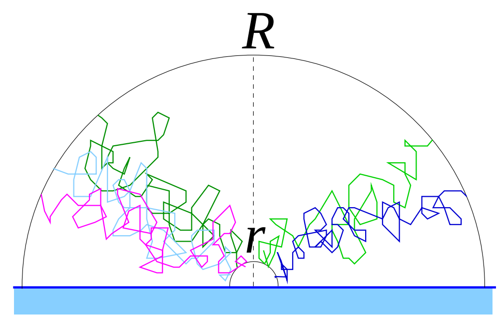
Original case of L mutually-avoiding simple RW's:

$n_1 = ... = n_L = 1.$

In the annulus $\mathbb{D}(r, R)$ in $\mathbb{C}$:

$$P_{n_1,\cdots,n_L}(r) \approx (r/R)^{2\zeta(n_1,\cdots,n_L)}$$

Boundary Case



Two mutually-avoiding packets of $n_1 = 3$, *and* $n_2 = 2$ *independent random walks, in the half-plane* $\mathbb{H}$.

Probability near a Dirichlet boundary

$$\tilde{P}_{n_1,\cdots,n_L}(t) \approx t^{-\frac{1}{2}\tilde{\zeta}(n_1,\cdots,n_L)},$$

and for crossing the half-annulus $\tilde{\mathbb{D}}(r,R)$ in $\mathbb{H}$

$$\tilde{P}_{n_1,\cdots,n_L}(r) \approx (r/R)^{\tilde{\zeta}(n_1,\cdots,n_L)}.$$

Cascade Relations

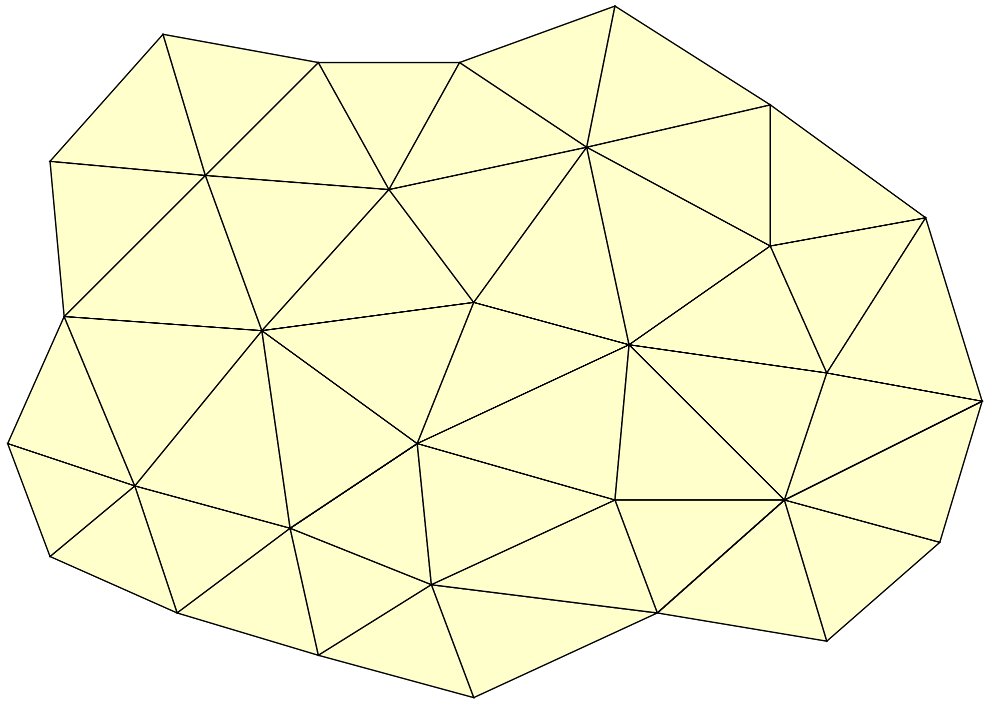
$$\begin{cases} \tilde{\zeta}(n_1, \cdots, n_L) = U\left(\sum_{l=1}^{L} U^{-1}(n_l)\right) \\ \zeta(n_1, \cdots, n_L) = V\left(\sum_{l=1}^{L} U^{-1}(n_l)\right) \end{cases}$$

$$\begin{cases} U(L) = \tilde{\zeta}_L \quad \left\{= \frac{1}{3}L\,(1+2L)\right\} \\ V(L) = \zeta_L \quad \left\{= \frac{1}{24}(4L^2-1) = U\left[\frac{1}{2}\left(L-\frac{1}{2}\right)\right]\right\} \\ \\ U^{-1}(n) = \frac{1}{4}(\sqrt{24n+1}-1) \end{cases}$$

- Lawler & Werner (98): Conformal invariance of Brownian motions
- B.D. (98): Interpretation and calculation in terms of "Quantum Gravity"

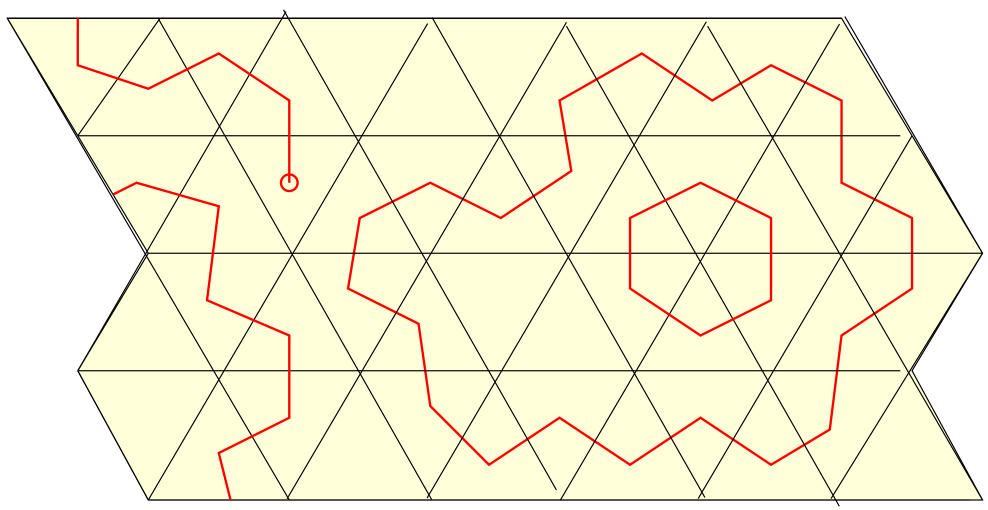
2D Quantum Gravity

Randomly Triangulated Lattice



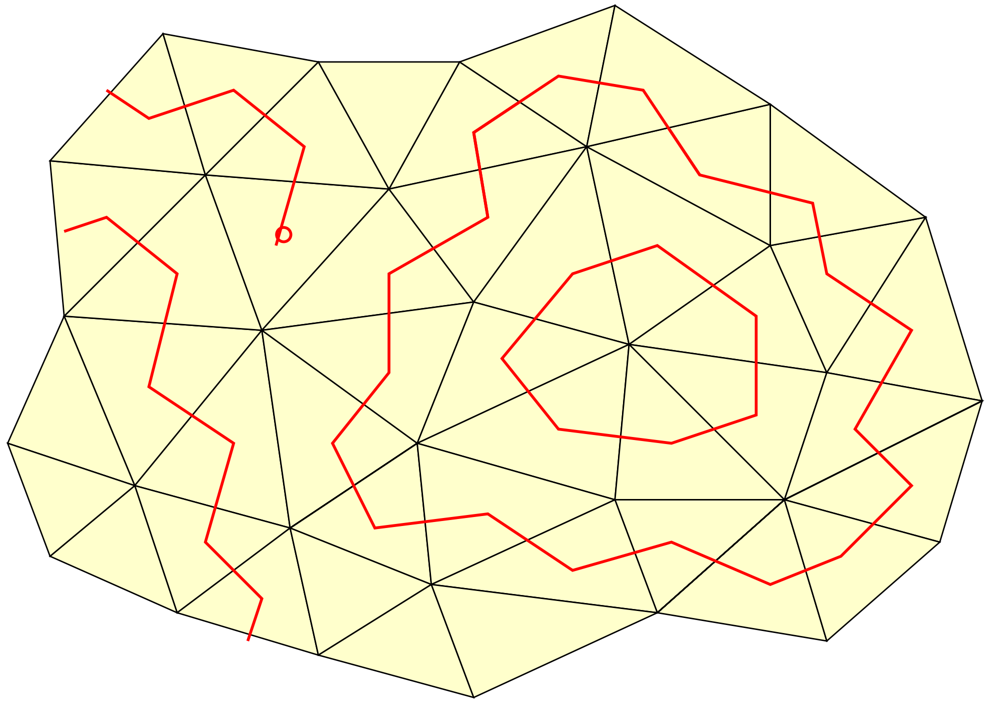
A random planar triangular lattice.

Statistical Mechanics on a Regular Lattice



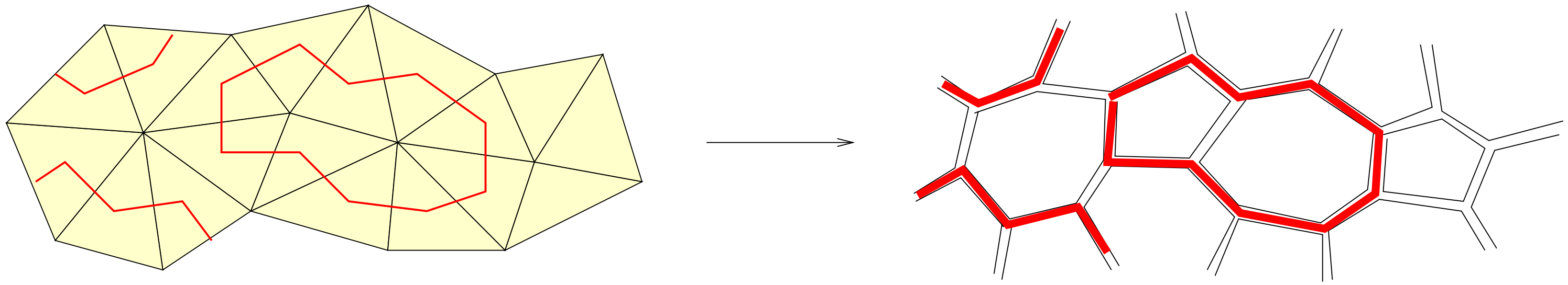
Random lines on the (dual of) a regular triangular lattice

Statistical Mechanics on a Random Lattice



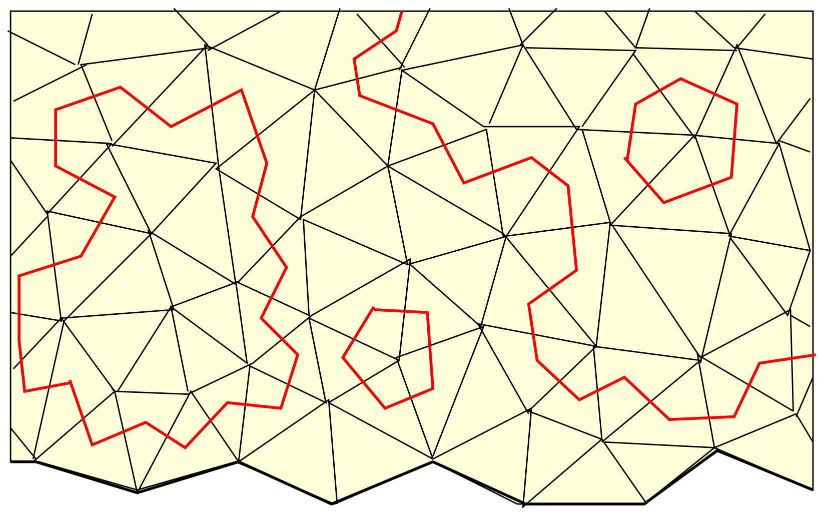
Random lines on a random planar triangular lattice

Dual Lattice



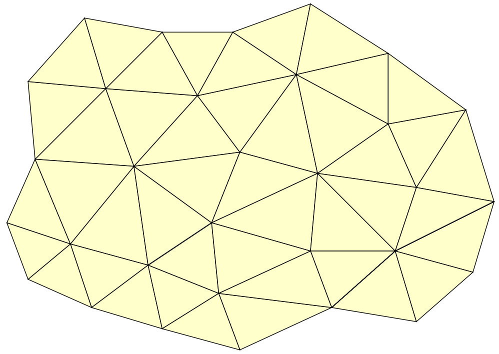
Random loops on the dual random lattice

Boundary Effects



Dirichlet boundary conditions on a random disk

Partition Function



Random planar triangular lattice G with fixed spherical topology.

$$Z(\beta) = \sum_{\text{planar } G} \frac{1}{S(G)} e^{-\beta|G|},$$

β: 'chemical potential' for the area, i.e., number of vertices $|G|$ of G; $S(G)$ its symmetry factor. Any fixed Euler characteristic χ possible; here $\chi = 2$.

Critical Behavior

The partition sum converges for β larger than some critical β_c. For $\beta \to \beta_c^+$ a singularity appears due to infinite graphs

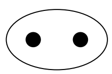
$$Z(\beta, \chi) \sim (\beta - \beta_c)^{2-\gamma_{\mathrm{str}}(\chi)},$$

where $\gamma_{\mathrm{str}}(\chi)$ is the string susceptibility exponent, depending on the genus of G through its Euler characteristic χ. For pure gravity and for the spherical topology

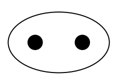
$$\gamma_{\mathrm{str}}(\chi = 2) = -\frac{1}{2}.$$

Doubly Punctured Sphere

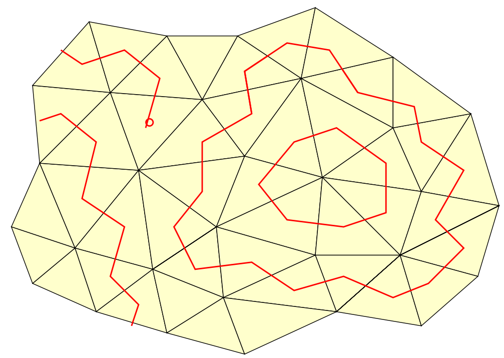
A particular partition function plays an important role, that of the doubly punctured sphere:

$Z[$  $] := \dfrac{\partial^2}{\partial\beta^2} Z(\beta, \chi = 2) = \sum_{G(\chi=2)} \dfrac{1}{S(G)} |G|^2 e^{-\beta|G|},$

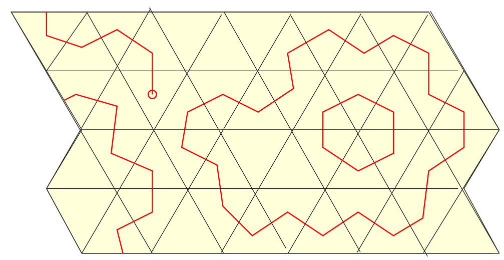
scaling as

$Z[$  $] \sim (\beta - \beta_c)^{-\gamma_{\text{str}}(\chi=2)}.$

KPZ *Knizhnik, Polyakov, Zamolodchikov, 88*



A “conformal operator” O (e.g. creating the line extremity) has conformal weight Δ (or $\tilde{\Delta}$) in (boundary) quantum gravity.



The same operator has conformal weight $\zeta = U(\Delta)$ in $\mathbb{C}$
($\tilde{\zeta} = U(\tilde{\Delta})$ in $\mathbb{H}$.)

KPZ

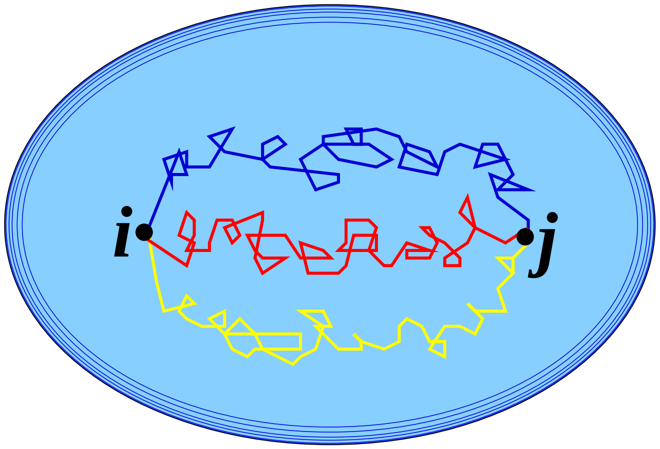
A fundamental quadratic relation exists between conformal weights Δ on a random planar surface (resp. $\tilde{\Delta}$ on a random disk) and those ζ in $\mathbb{C}$ (resp. $\tilde{\zeta}$ in $\mathbb{H}$)

$$\zeta = U(\Delta) = \Delta \frac{\Delta - \gamma}{1 - \gamma},$$

with γ the string susceptibility exponent. For Brownian paths, self-avoiding walks, and percolation, $\gamma = -1/2$, and the KPZ relation becomes

$$\zeta = U(\Delta) = \frac{1}{3}\Delta(1 + 2\Delta).$$

Random Walks on a Random Lattice



Set of $L = 3$ *mutually-avoiding random walks*

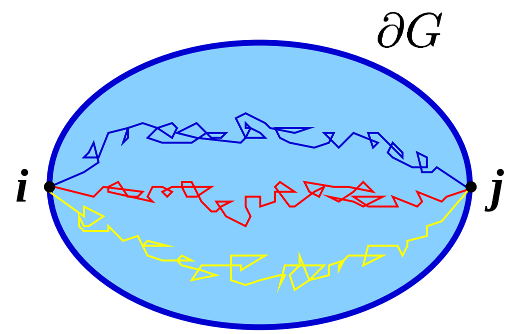
Walk set $\mathcal{B} = \{B_{ij}^{(l)}, l = 1, ..., L\}$ on the random planar graph G, started at vertex $i \in G$, ended at vertex $j \in G$.

Random Walk Partition Function

$$Z_L(\beta, z) = \sum_{\text{planar } G} \frac{1}{S(G)} e^{-\beta|G|} \sum_{i,j \in G} \sum_{\substack{B_{ij}^{(l)} \\ l=1,\ldots,L}} z^{|\mathcal{B}|},$$

where a “fugacity” z is associated with the total number $|\mathcal{B}| = \left| \cup_{l=1}^{L} B^{(l)} \right|$ of vertices visited by the walks.

Boundary Partition Function



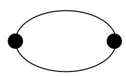
$L = 3$ *mutually-avoiding RW's traversing a random disk.*

Boundary case: G has the disk topology and the random walks connect sites i and j on the boundary ∂G, with fugacity $e^{-\tilde{\beta}}$ for the boundary's length $|\partial G|$

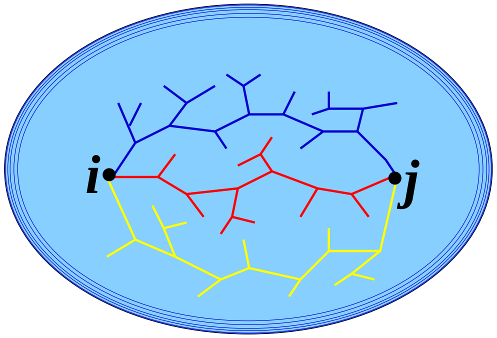
$$\tilde{Z}_L(\beta, \tilde{\beta}, z) = \sum_{\text{disk } G} e^{-\beta|G|} e^{-\tilde{\beta}|\partial G|} \sum_{i,j \in \partial G} \sum_{\substack{B_{ij}^{(l)} \\ l=1,\dots,L}} z^{|\mathcal{B}|},$$

Punctured Disk Partition Function

Partition function of the disk with two boundary punctures: it corresponds to the $L = 0$ case of the $\tilde{Z}_L$'s

$Z($  $) = \tilde{Z}_{L=0}(\beta, \tilde{\beta}) = \sum_{\text{disk } G} e^{-\beta|G|} e^{-\tilde{\beta}|\partial G|} \, |\partial G|^2 .$

Equivalent Random Trees *(Aldous-Broder)*

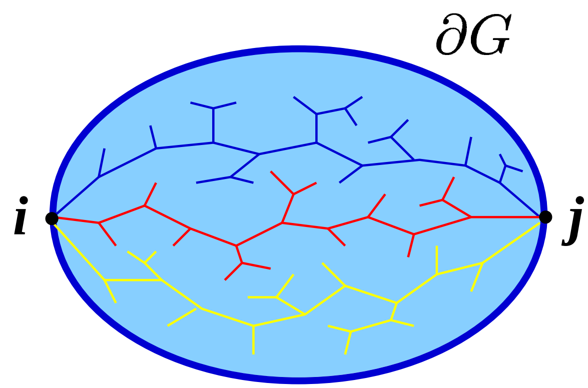


$L-$tree partition function on the random lattice:

$$Z_L(\beta, z) = \sum_{\text{planar } G} \frac{1}{S(G)} e^{-\beta|G|} \sum_{i,j \in G} \sum_{\substack{T_{ij}^{(l)} \\ l=1,\ldots,L}} z^{|T|},$$

$\left\{T_{ij}^{(l)}, l = 1, \cdots, L\right\}$ are L mutually-avoiding trees, joining sites i and j; a fugacity z governs the total number of tree vertices $|T| = \left|\cup_{l=1}^{L} T^{(l)}\right|$.

Boundary Trees

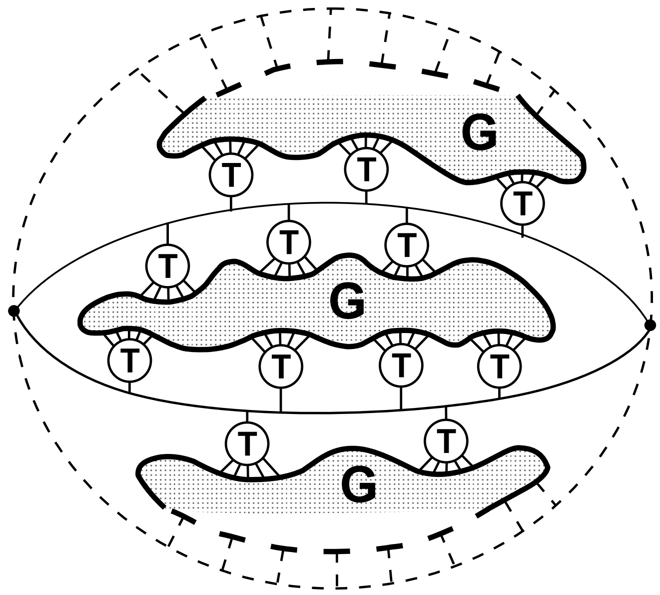


$L = 3$ *mutually-avoiding random trees traversing a random disk*

Boundary case where G is a disk and the trees connect sites i and j on the boundary ∂G, with a fugacity $\tilde{z} = \exp(-\tilde{\beta})$ associated with the boundary's length

$$\tilde{Z}_L(\beta, z, \tilde{z}) = \sum_{\text{disk } G} e^{-\beta|G|} e^{-\tilde{\beta}|\partial G|} \sum_{i,j \in \partial G} \sum_{\substack{T_{ij}^{(l)} \\ l=1,\ldots,L}} z^{|T|}$$

Quantum Surgery



The shaded areas are portions of random lattice G with a disk topology; $L = 2$ trees connect the end-points. Each corresponds to a generating function, as follows. *(For a global disk topology, the dashed lines represent the boundary, whereas for the sphere the top and bottom dashed lines are identified)*

Tree Generating Function

Each random tree has a generating function

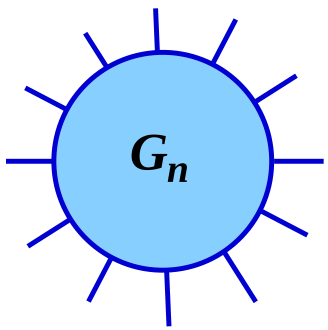
$$T(x) = \sum_{n \geqslant 1} x^n T_n,$$

where $T_1 \equiv 1$ and T_n is the number of *rooted* planar trees with n external vertices (excluding the root):

$$T(x) = \frac{1}{2}(1 - \sqrt{1-4x}).$$

The patches of random lattice are representated as follows.

Disk Generating Function



A planar random disk with n external legs

Partition function of a random disk with n external vertices:

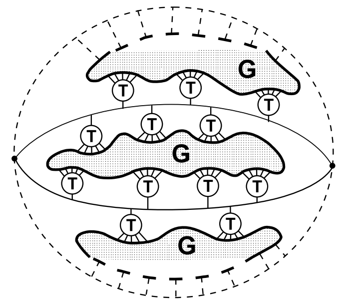
$$G_n(\beta) = \sum_{n-\text{leg planar } G} e^{-\beta|G|}.$$

Large$-N$ limit of a random $N \times N$ matrix integral:

$$G_n(\beta) = \int_a^b d\lambda\, \rho(\beta, \lambda)\, \lambda^n,$$

$\rho(\beta, \lambda)$: spectral eigenvalue density, with compact support $[a(\beta), b(\beta)]$.

Integral Representation



$$Z_L(\beta, z) = \int_a^b \prod_{l=1}^{L} d\lambda_l \, \rho(\beta, \lambda_l) \prod_{l=1}^{L} \mathcal{T}(z\lambda_l, z\lambda_{l+1}),$$

with a *cyclic* structure $\lambda_{L+1} \equiv \lambda_1$. The disk G_l of random surface between trees $T^{(l-1)}$, $T^{(l)}$ contributes a spectral density $\rho(\lambda_l)$. The backbone of tree $T^{(l)}$ between disks G_l and G_{l+1} yields a "propagator" $\mathcal{T}(z\lambda_l, z\lambda_{l+1})$

$$\mathcal{T}(x,y) := [1 - T(x) - T(y)]^{-1}.$$

Boundary Integral Representation

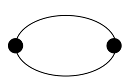
Boundary partition function:

$$\tilde{Z}_L(\beta, z, \tilde{z}) \;=\; \int_a^b \prod_{l=1}^{L+1} d\lambda_l \, \rho(\beta, \lambda_l) \prod_{l=1}^{L} \mathcal{T}(z\lambda_l, z\lambda_{l+1})$$
$$\times \mathcal{L}(\tilde{z}\lambda_1)\,\mathcal{L}(\tilde{z}\lambda_{L+1})$$

with two extra propagators $\mathcal{L}$ describing the two boundary lines:

$$\mathcal{L}(\tilde{z}\lambda) := (1 - \tilde{z}\lambda)^{-1}.$$

This gives for the two-puncture disk partition function

$Z($  $)$

$$= \tilde{Z}_{L=0}(\beta, \tilde{z}) = \int_a^b d\lambda \, \rho(\beta, \lambda)\, \mathcal{L}^2(\tilde{z}\lambda).$$

Critical Behavior

Critical behavior of $Z_L(\beta, z)$ or $\tilde{Z}_L\left(\beta, z, \tilde{z} = \exp(-\tilde{\beta})\right)$:

Triple scaling limit: $\beta \to \beta_c^+$ *(infinite random lattice)*, $\tilde{\beta} \to \tilde{\beta}_c^+$ *(infinite boundary length)*, and $z \to z_c^-$ *(infinite RW's)*; the average lattice area, boundary length, and RW's sizes respectively scale as

$$\langle |G| \rangle \sim (\beta - \beta_c)^{-1}, \langle |\partial G| \rangle \sim (\tilde{\beta} - \tilde{\beta}_c)^{-1}, \langle |\mathcal{B}| \rangle \sim (z_c - z)^{-1}.$$

The later analysis of the singular behavior in terms of "conformal weights" requires a natural *finite-size scaling* (hereafter dropping $\langle \cdots \rangle$)

$$|\partial G| \sim |G|^{1/2} \sim |\mathcal{B}|.$$

Power Counting

Each component of the integrals scales with a power law of the mean area $\langle |G| \rangle$:

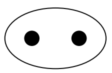
$$
\begin{aligned}
Z_L &\sim \left(\int \rho d\lambda \star \mathcal{T} \right)^L \\
\tilde{Z}_L &\sim \left(\int \rho d\lambda \star \mathcal{T} \right)^L \star \int \rho d\lambda \star \mathcal{L}^2 \\
Z(\bullet\!\!\bigcirc\!\!\bullet) &= \tilde{Z}_0 \sim \int \rho d\lambda \star \mathcal{L}^2
\end{aligned}
$$

where the $\star$ symbolic notation represents the factorisation of scaling behaviors. This implies the fundamental scaling relations:

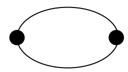
$$
\begin{aligned}
Z_L &\sim (Z_1)^L \\
&\sim \frac{\tilde{Z}_L}{Z(\bullet\!\!\bigcirc\!\!\bullet)} \sim \left[\frac{\tilde{Z}_1}{Z(\bullet\!\!\bigcirc\!\!\bullet)} \right]^L .
\end{aligned}
$$

Conformal Weights

The partition function Z_L represents a doubly punctured sphere with two *conformal operators*, of conformal weights Δ_L (here two vertices sources of L mutually-avoiding RW's):

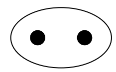
$$Z_L \sim Z[$$  $$] \star |G|^{-2\Delta_L} \,.$$

The boundary partition function $\tilde{Z}_L$ corresponds to a doubly punctured disk with two *boundary operators* of conformal weights $\tilde{\Delta}_L$:

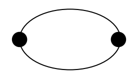
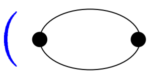
$$\tilde{Z}_L \sim Z($$  $$) \star |\partial G|^{-2\tilde{\Delta}_L} \,.$$

Structural Relations

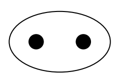
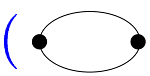
- Doubly punctured sphere partition function [$\gamma := \gamma_{\rm str}(\chi = 2)$]:

$$Z[$$  $$] \sim (\beta - \beta_c)^{-\gamma} \sim |G|^{\gamma}.$$

- Scaling equivalences for *bulk* and *boundary* partition functions:

$$Z_L \sim (Z_1)^L \sim \tilde{Z}_L / Z($$  $$) \sim \left[\tilde{Z}_1 / Z($$  $$)\right]^L.$$

- Defi nitions of conformal weights

$$Z_L \sim Z[$$  $$] \star |G|^{-2\Delta_L}, \quad \tilde{Z}_L / Z($$  $$) \sim |\partial G|^{-2\tilde{\Delta}_L}$$

$$\Rightarrow Z_L \sim |G|^{\gamma - 2\Delta_L} \sim |\partial G|^{-2\tilde{\Delta}_L} \sim (Z_1)^L.$$

- Perimeter-area scaling $|\partial G| \sim |G|^{1/2}$

$$\Rightarrow 2\Delta_L - \gamma = \tilde{\Delta}_L = L\,\tilde{\Delta}_1.$$

- BULK $\Longleftrightarrow$ BOUNDARY • LINEARITY OF BOUNDARY WEIGHTS

Brownian Exponents in Q G

The analysis of the singularities of the integrals gives

$$
\begin{aligned}
2\Delta_L - \gamma &= \tilde{\Delta}_L = L\,\tilde{\Delta}_1 \\
\tilde{\Delta}_1 &= 1.
\end{aligned}
$$

From $\gamma = -\frac{1}{2}$ of pure gravity, one finally gets

$$
\begin{aligned}
\Delta_L &= \frac{1}{2}\left(L - \frac{1}{2}\right) \\
\tilde{\Delta}_L &= L.
\end{aligned}
$$

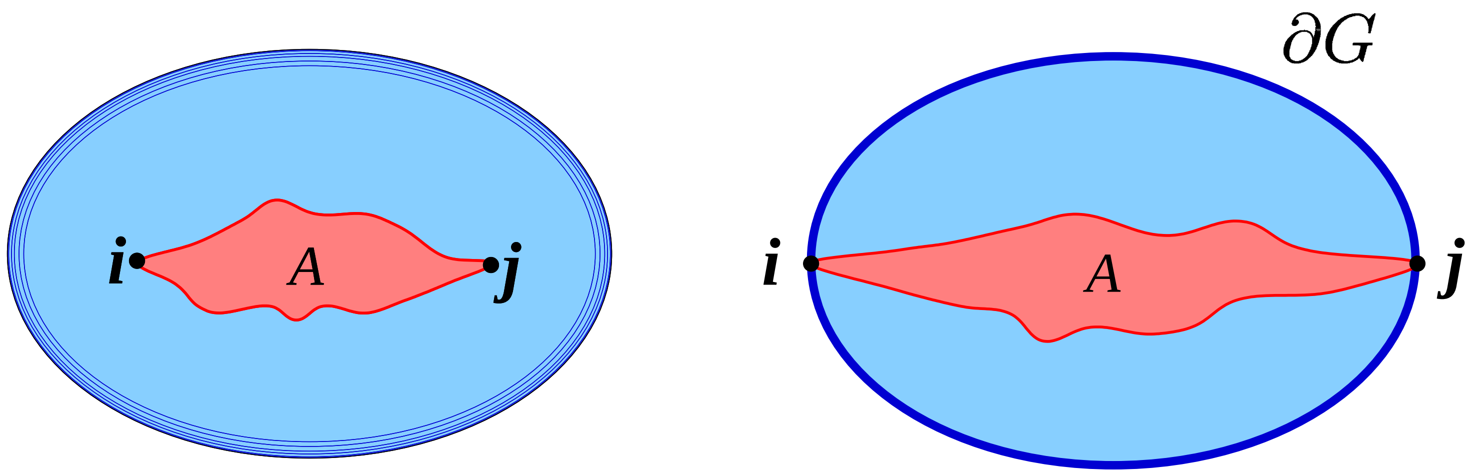
Exponents in $\mathbb{C}$ or $\mathbb{H}$ & KPZ

$$U(\Delta) = \frac{1}{3}\Delta(1+2\Delta)$$

$$\begin{cases} \Delta_L = \frac{1}{2}\left(L - \frac{1}{2}\right), & \zeta_L = U(\Delta_L) = \frac{1}{24}\left(4L^2 - 1\right) \\ \tilde{\Delta}_L = L, & \tilde{\zeta}_L = U(\tilde{\Delta}_L) = \frac{1}{3}L(1+2L) \end{cases} \quad \mathcal{QED}$$

Life in QG is easy

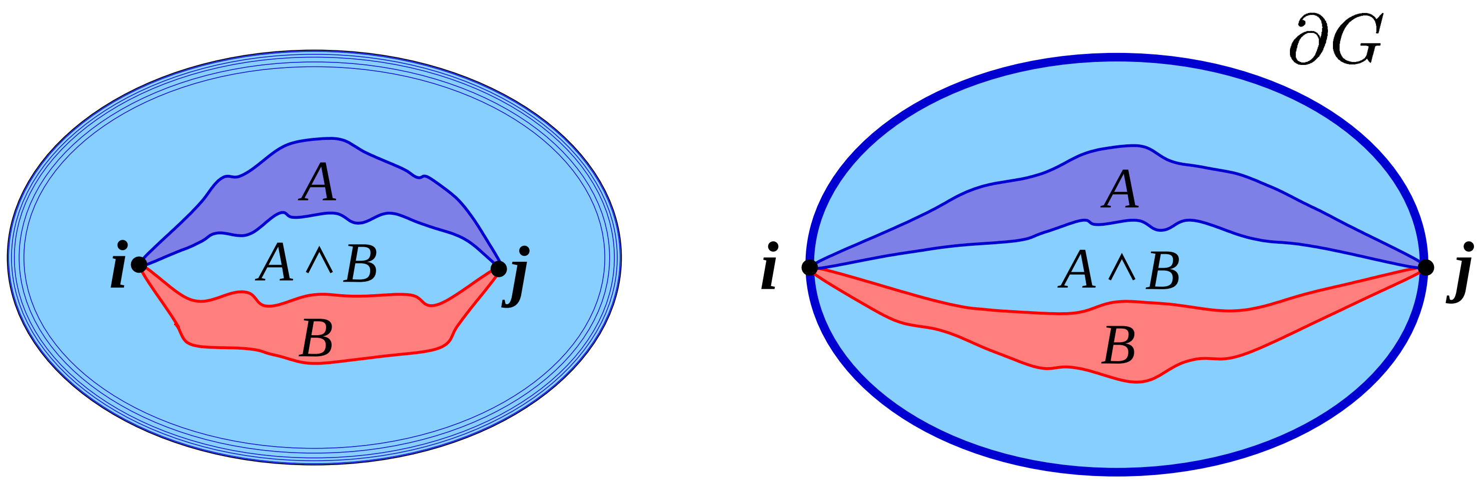
Bulk-Boundary Relation



Δ $\tilde{\Delta}$

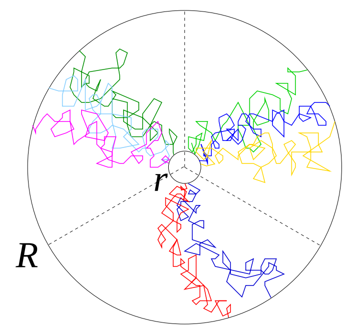
$$2\Delta - \gamma = \tilde{\Delta}$$

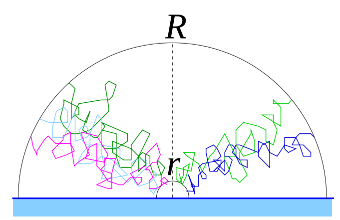
Quantum Boundary Additivity & Mutual Avoidance



$$2\Delta_{A\wedge B} - \gamma = \tilde{\Delta}_{A\wedge B} = \tilde{\Delta}_A + \tilde{\Delta}_B$$

Quantum Gravity & Packets of Walks





$$2\Delta\left\{n_1,\cdots,n_L\right\}+\frac{1}{2} = \tilde{\Delta}\{n_1,\cdots,n_L\}$$
$$= \sum_{l=1}^{L}\tilde{\Delta}(n_l).$$

Brownian Packet in Q G

Boundary conformal weight in $\mathbb{H}$ of a packet of n independent Brownian paths:

$$\tilde{\zeta} = n$$

Inverting KPZ:

$$\tilde{\Delta}(n) = U^{-1}(n) = \frac{1}{4}(\sqrt{24n+1} - 1).$$

The Brownian paths, independent in a fixed metric, are strongly coupled by the metric fluctuations in quantum gravity.

Back to the (Half-) Plane with KPZ

$$\begin{cases} \tilde{\xi}(n_1,\cdots,n_L) = U\left(\tilde{\Delta}\{n_1,\cdots,n_L\}\right) \\ \\ \zeta(n_1,\cdots,n_L) = V\left(\tilde{\Delta}\{n_1,\cdots,n_L\}\right) \\ \\ \tilde{\Delta}\{n_1,\cdots,n_L\} = \sum_{l=1}^{L} U^{-1}(n_l) = \sum_{l=1}^{L} \frac{1}{4}(\sqrt{24n_l+1}-1) \end{cases}$$

$$\begin{cases} U(\Delta) = \frac{1}{3}\Delta\left(1+2\Delta\right) \\ \\ V(\Delta) = U\left[\frac{1}{2}\left(\Delta - \frac{1}{2}\right)\right] = \frac{1}{24}(4\Delta^2 - 1), \end{cases}$$

Quantum gravity & cascade relations, $\mathcal{QED}$.

Mandelbrot Conjecture

$$\zeta(n=2) = V\left(U^{-1}(2)\right) = V\left(\frac{3}{2}\right)$$
$$= \zeta_{L=\frac{3}{2}} = \frac{1}{3}.$$

whence

$$D_{\text{Brown.Fr.}} = 2 - 2\zeta = \frac{4}{3}, \quad \mathcal{QED}$$

(*LSW, 2000*)