CONFORMAL RANDOM GEOMETRY & QUANTUM GRAVITY

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Mark Kac Seminar

on Probability and Physics

Utrecht

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CONFORMAL RANDOM GEOMETRY
&
MULTIFRACTALITY

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Service de Physique Théorique de Saclay

Mark Kac Seminar III

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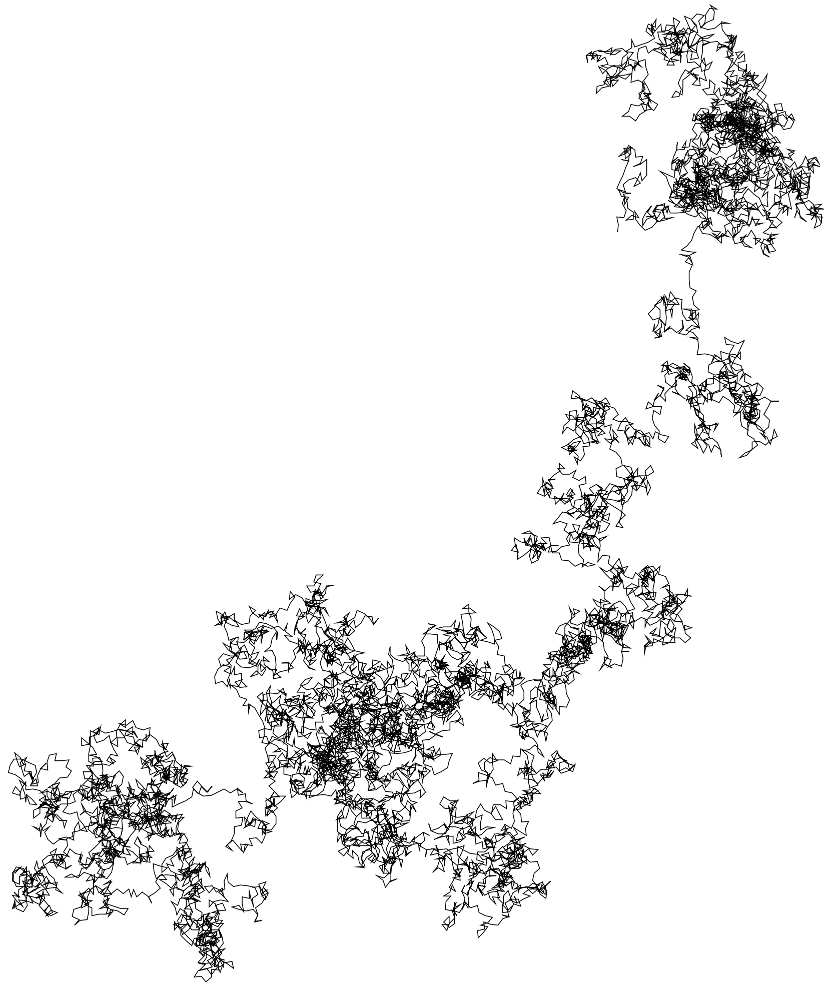
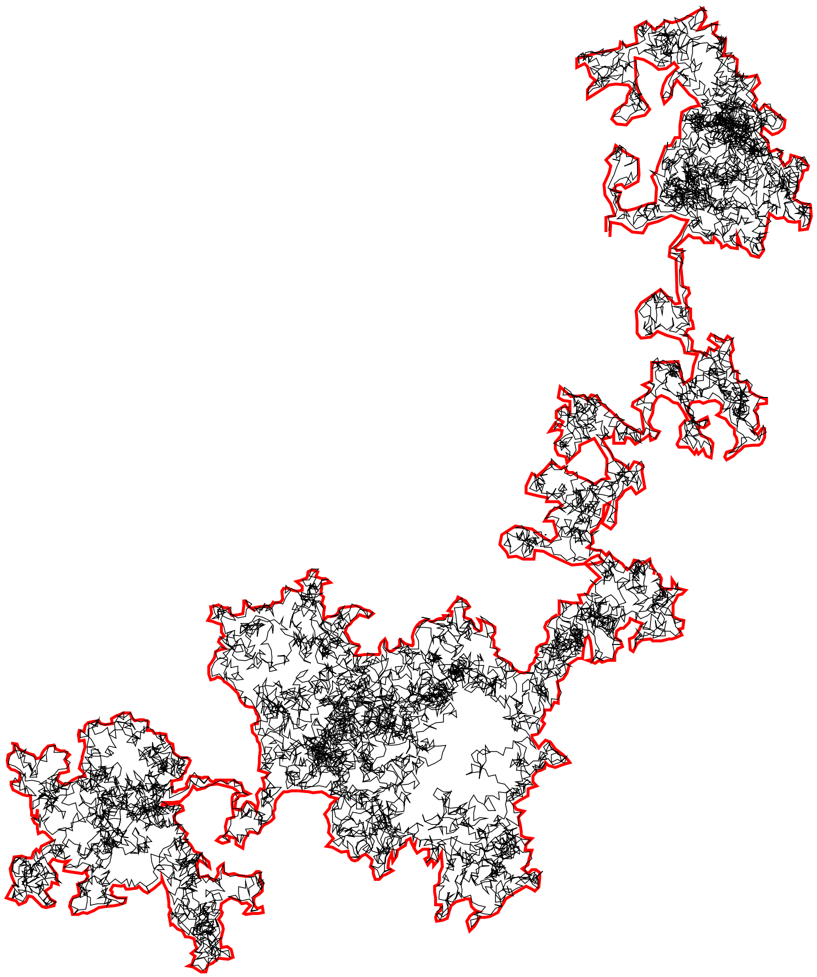
CONFORMALLY INVARIANT RANDOM SCALING CURVES

Brownian Path

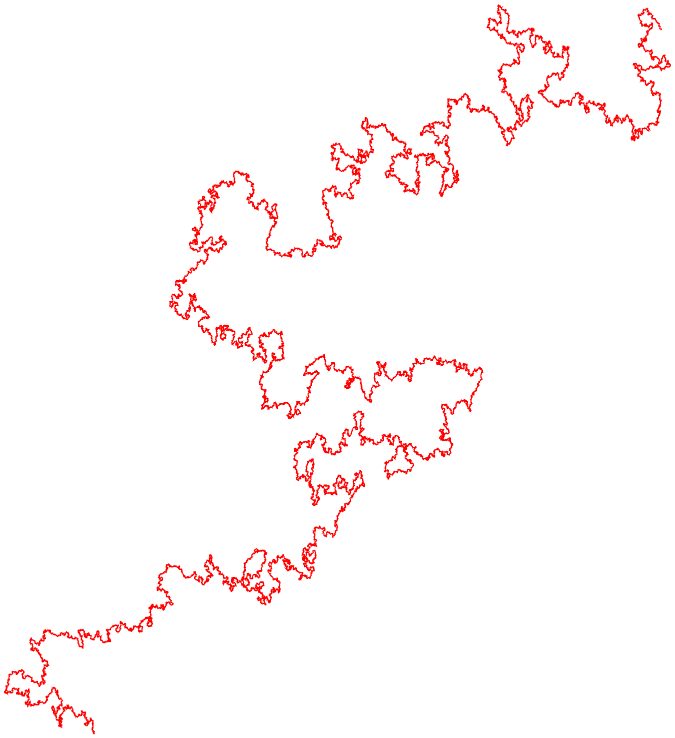


Brownian Frontier



Mandelbrot conjecture: $D = \frac{4}{3}$.

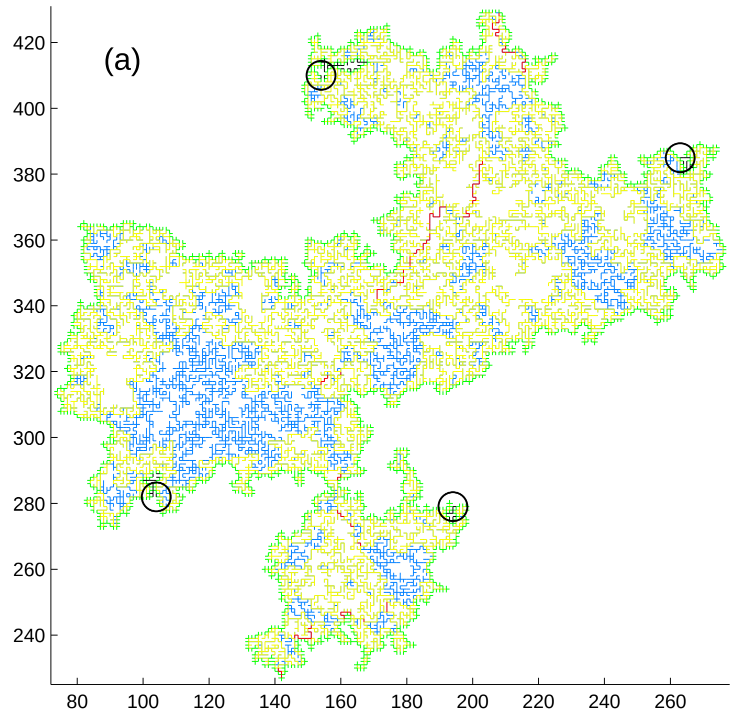
Self-Avoiding Walk



(courtesy of T. Kennedy)

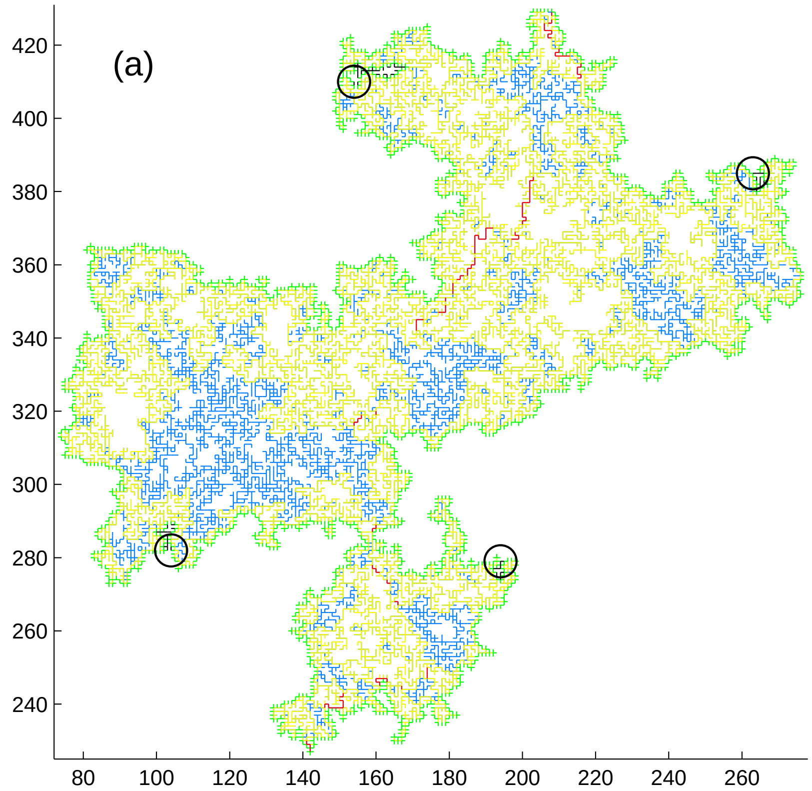
B. Nienhuis (1982): $D = \frac{4}{3}$

Percolation Hull & Frontier



Cluster; Hull: $D_{\mathrm{Hull}} = \frac{7}{4}$ *(DS, 1987; Smirnov; Beffara); External Perimeter:* $D_{\mathrm{EP}} = \frac{4}{3}$ *(ADA, 1999); (courtesy of J. Asikainen, et al., 2003).*

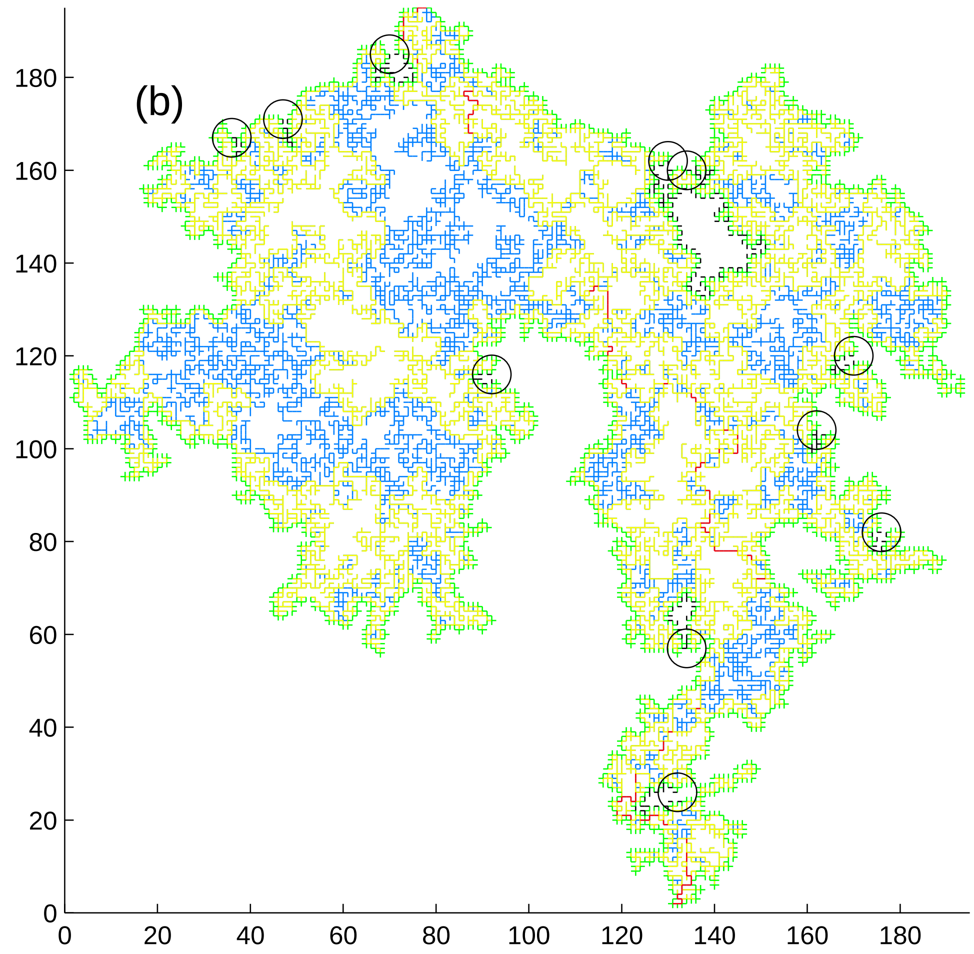
Percolation Cluster



Cluster; *hull*; *EP bonds*; *singly connected bonds*; ⊙ *gates to fjords.*

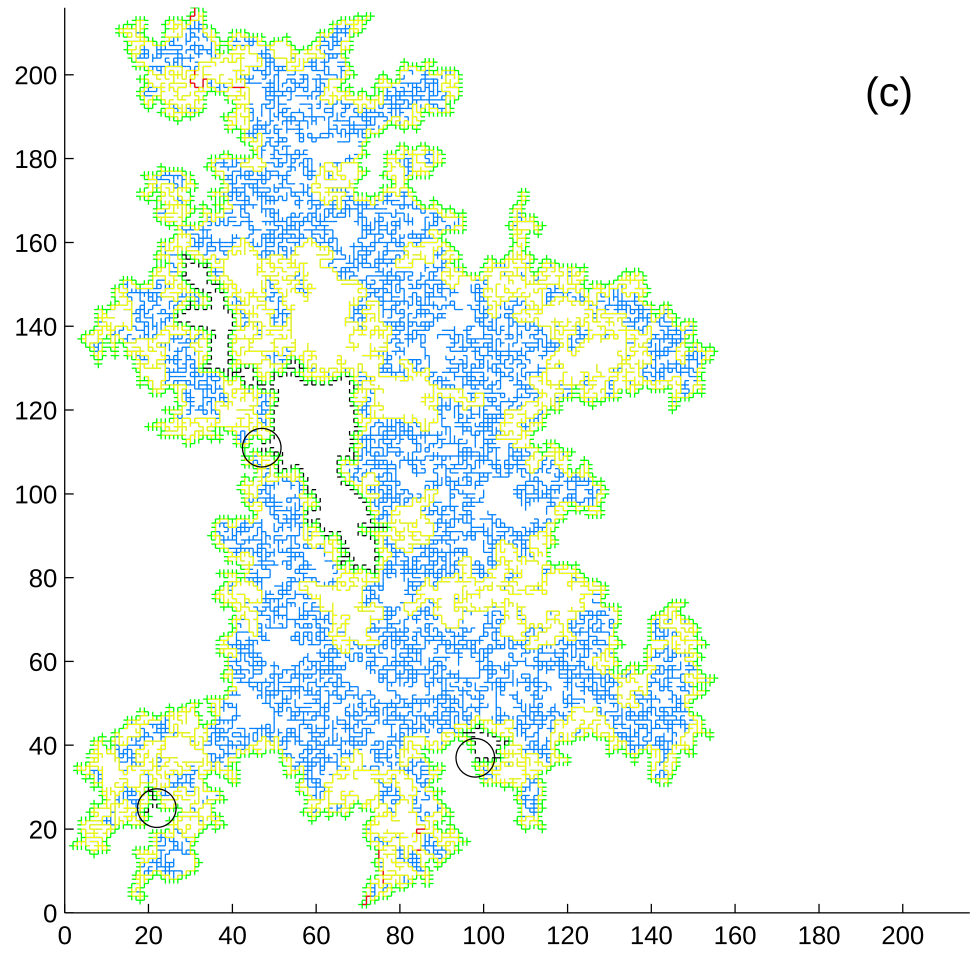
(J. Asikainen, A. Aharony, B.B. Mandelbrot, E. Rausch and J.-P. Hovi, Eur. Phys. J. B **34***, 479-487 (2003))*

Potts Cluster ($Q = 2$)



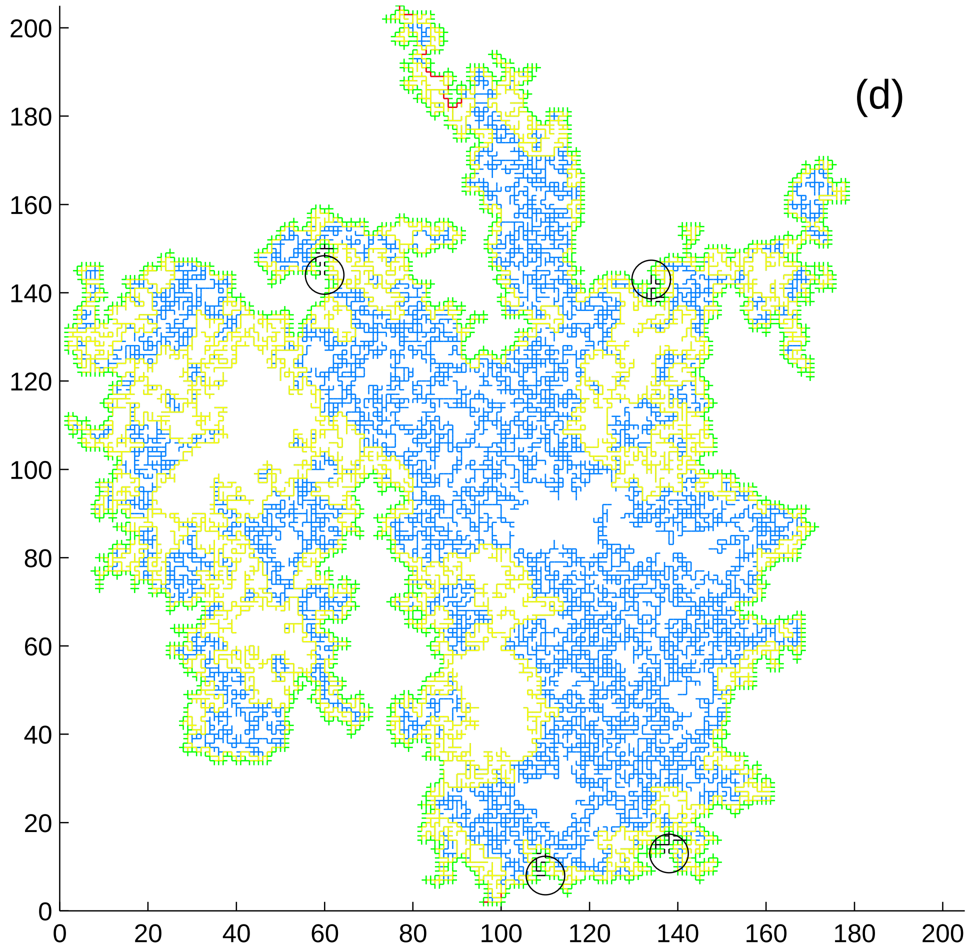
Cluster; *hull*; *EP bonds*; *singly connected bonds*; ⊙ *gates to fjords.*

Potts Cluster ($Q = 3$)



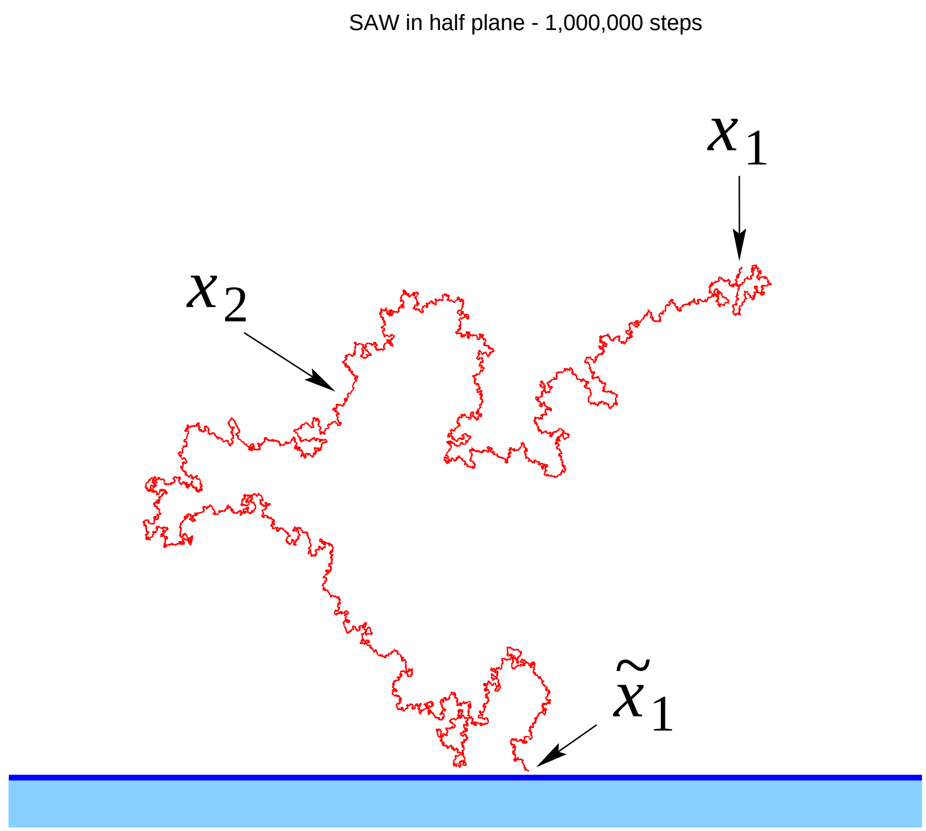
Cluster; *hull*; *EP bonds*; *singly connected bonds*; ⊙ *gates to fjords.*

Potts Cluster $(Q = 4)$



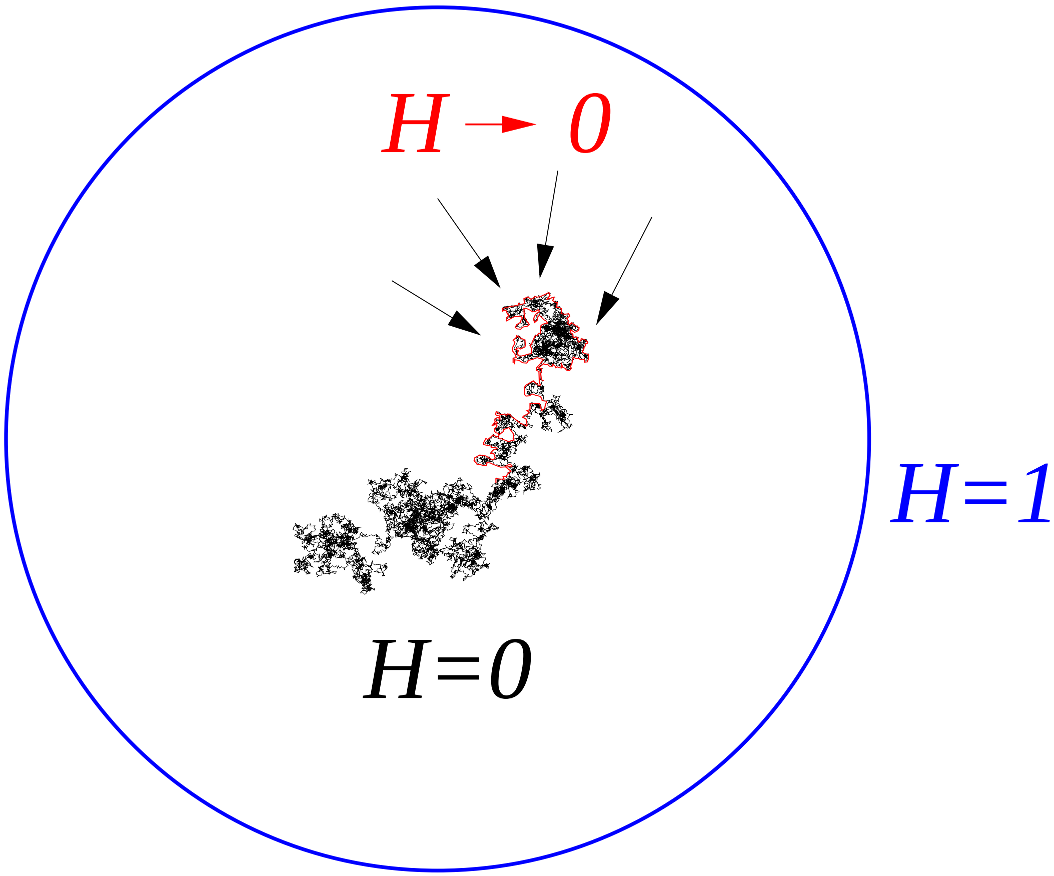
Cluster; *hull*; *EP bonds*; *singly connected bonds*; $\odot$ *gates to fjords.*

SLE_κ *(Schramm, 1999)*

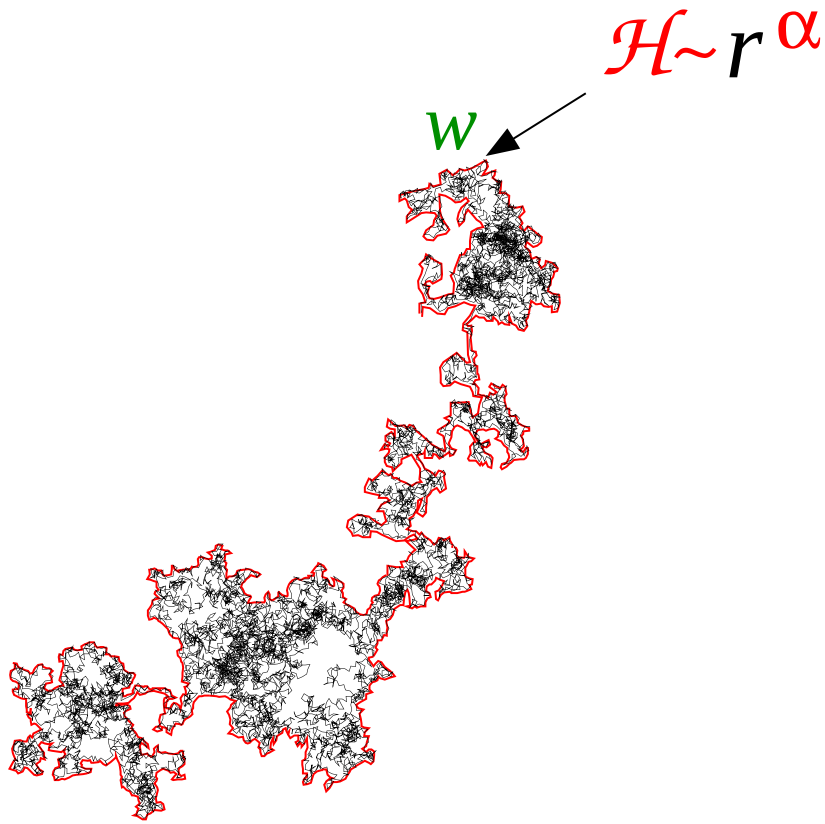


(G. Lawler, O. Schramm & W. Werner; S. Rohde & O. S.; S. Smirnov; M. Bauer & D. Bernard; J. Cardy; W. Kager & B. Nienhuis)

Potential Distribution

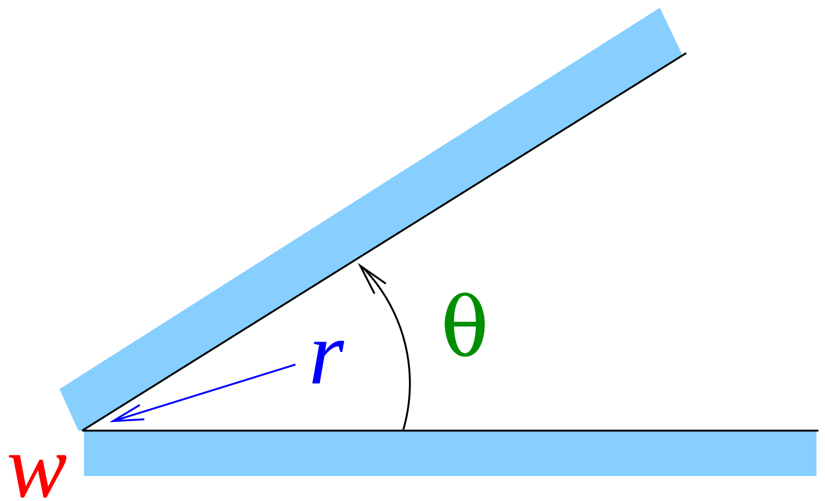


Multifractality



$$w \in \mathcal{F}_{\alpha}: \quad \dim \mathcal{F}_{\alpha} = f(\alpha)$$

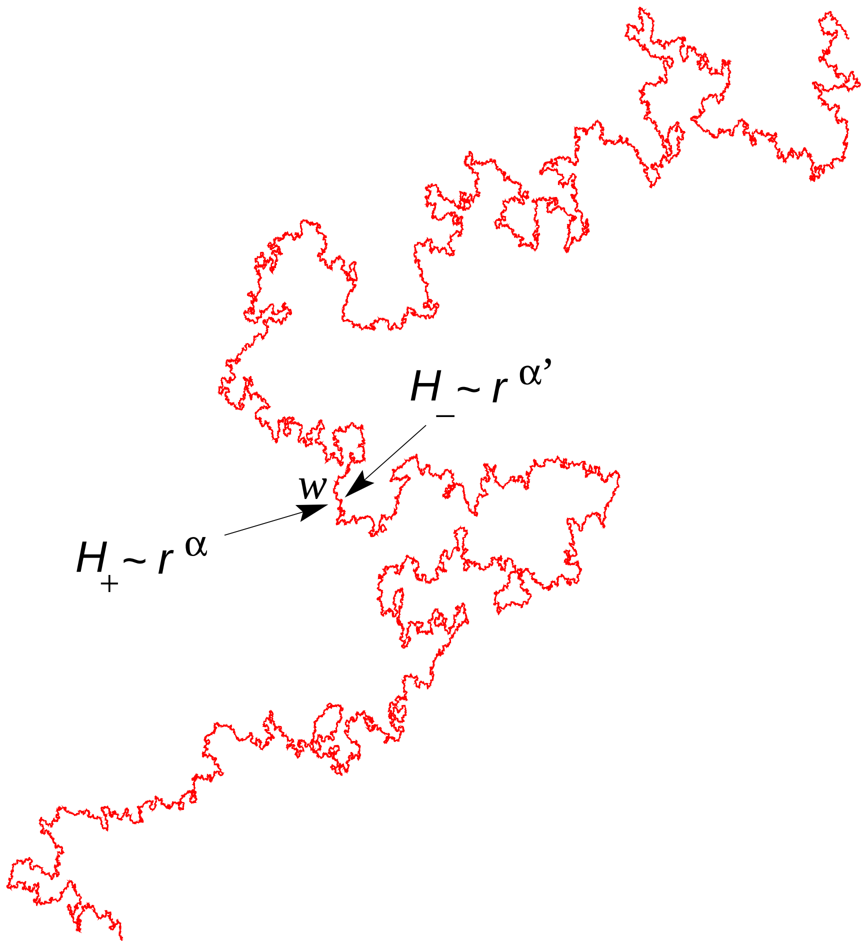
Electrostatic Wedge



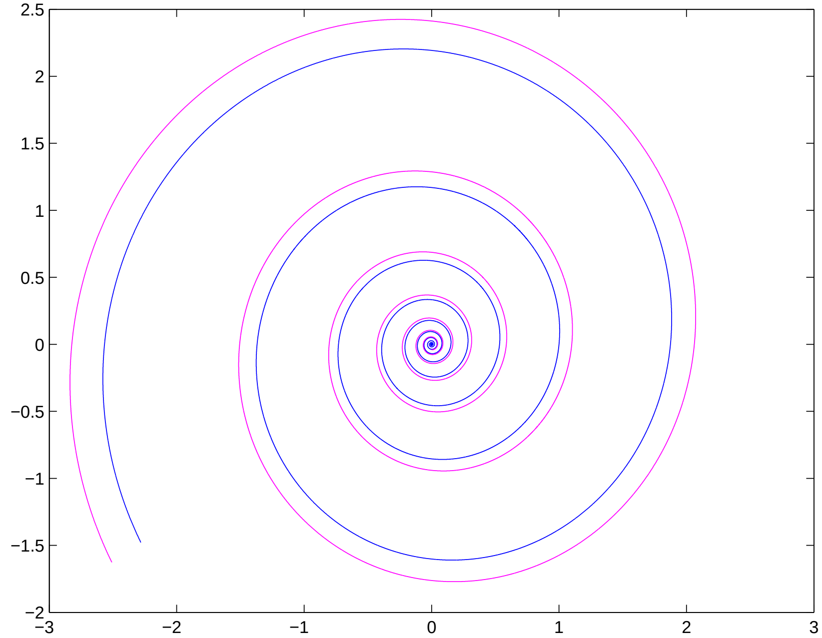
$$H(w,r) \sim r^{\pi/\theta}; \quad \alpha = \pi/\theta$$

$$\theta \in (0, 2\pi]; \quad \alpha \in \left[\frac{1}{2}, +\infty\right)$$

Double-Sided Potential



Logarithmic Spirals



A point w on the frontier with a double logarithmic spiral.

$$\varphi(w, r) = \lambda \ln r$$

Mixed Multifractal Spectrum

(Ilia Binder, 1996)

$$w \in \mathcal{F}_{\alpha,\lambda} \iff \begin{cases} H(w,r) \sim r^{\alpha} \\ \varphi(w,r) \sim \lambda \ln r \end{cases}$$

$$\dim \mathcal{F}_{\alpha,\lambda} = f(\alpha,\lambda)$$

Spectra Hierarchy

$$\begin{aligned} f(\alpha) &= \sup_{\lambda} f(\alpha, \lambda) \\ &= f(\alpha, \lambda = 0) \ \text{(by symmetry)} \end{aligned}$$

Rotation Dimensions

$$D_{\mathrm{EP}}(\lambda) = \sup_{\alpha} f(\alpha, \lambda)$$

Spectra Hierarchy

$$
\begin{aligned}
f(\alpha) &= \sup_{\lambda} f(\alpha, \lambda) \\
&= f(\alpha, \lambda = 0) \;\; (\text{by symmetry})
\end{aligned}
$$

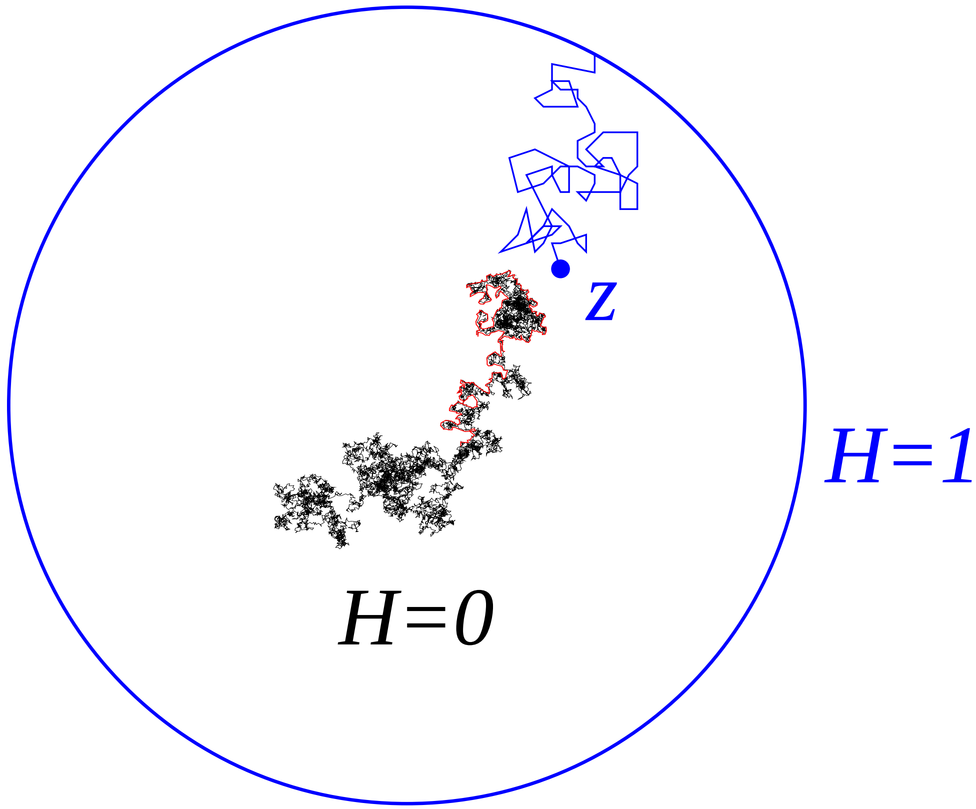
Rotation Dimensions

$$D_{\mathrm{EP}}(\lambda) = \sup_{\alpha} f(\alpha, \lambda)$$

External Perimeter Dimension

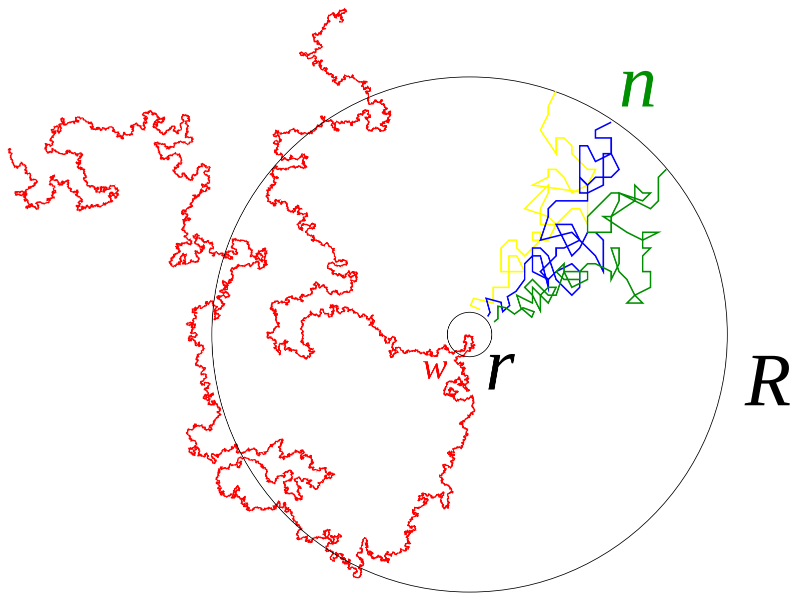
$$
\begin{aligned}
D_{\mathrm{EP}} &= \sup_{\alpha,\lambda} f(\alpha, \lambda) \\
&= \sup_{\alpha} f(\alpha, \lambda = 0) = \sup_{\alpha} f(\alpha) \\
&= \sup_{\lambda} D_{\mathrm{EP}}(\lambda)
\end{aligned}
$$

Harmonic Measure & Brownian Paths



Potential $H(z)$: Probability that the auxiliary Brownian path started at z hits 1 before 0 *(Kakutani, 1942)*.

Moments & Brownian Paths



$$\sum_{w} H^{n}(w,r) \approx (r/R)^{2x(n)-2}$$

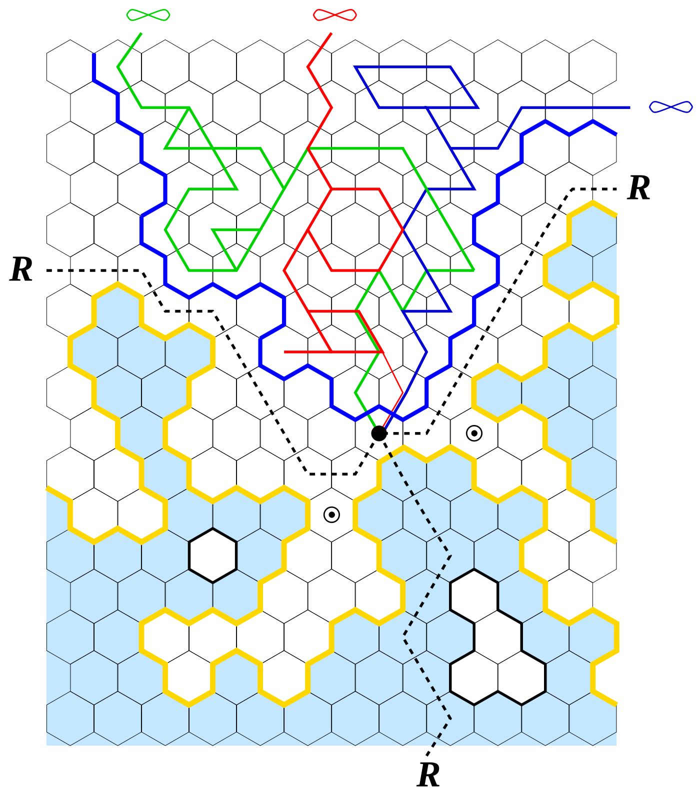
$H(w,r)$: harmonic measure in ball $B(w,r)$

Legendre Transform

Define $\tau(n) = 2x(n) - 2$

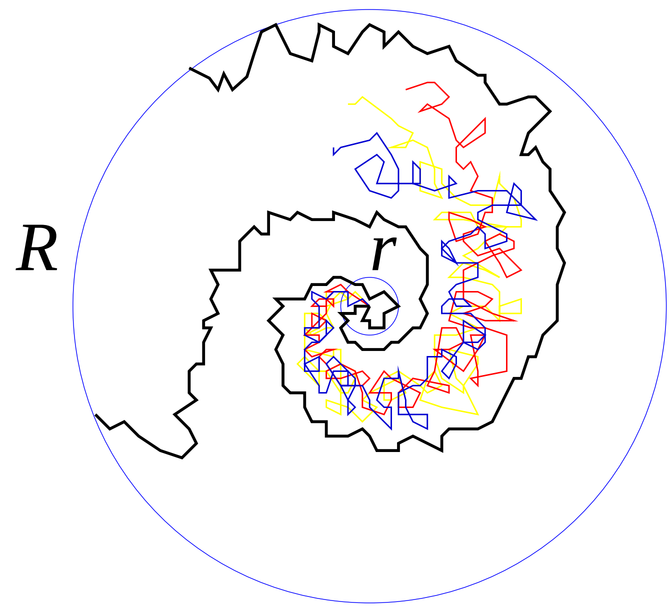
$$
\begin{aligned}
\alpha &= \frac{\partial \tau}{\partial n}(n) \\
\alpha n &= f(\alpha) + \tau(n) \\
n &= \frac{\partial f}{\partial \alpha}(\alpha).
\end{aligned}
$$

Diffusion & Percolation Frontier



An accessible site (●) *on the external perimeter in percolation, with three escape paths. The entrances of* fjords ⊙ *close in the scaling limit. Full hull dimension (gold):* $\frac{7}{4}$. *External perimeter dimension* $\frac{4}{3}$.

Mixed Moments & Random Walk Windings



$$Z_{n,p} = \sum_{w} H^{n}(w,r)\exp\left(p\,\varphi(w,r)\right) \approx (r/R)^{2x(n,p)-2}$$

$H(w,r)$: harmonic measure in ball $B(w,r)$

$\varphi(w,r)$: rotation angle

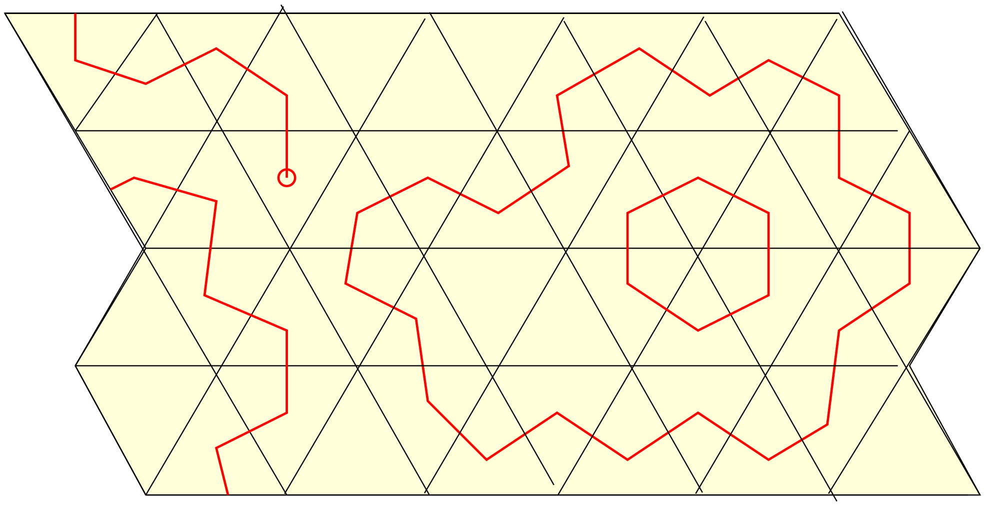
Double Legendre Transform

Define $\tau(n,p) = 2x(n,p) - 2$

$$\begin{aligned} \alpha &= \frac{\partial \tau}{\partial n}(n,p), \quad \lambda = \frac{\partial \tau}{\partial p}(n,p), \\ f(\alpha,\lambda) &= \alpha n + \lambda p - \tau(n,p), \\ n &= \frac{\partial f}{\partial \alpha}(\alpha,\lambda), \quad p = \frac{\partial f}{\partial \lambda}(\alpha,\lambda). \end{aligned}$$

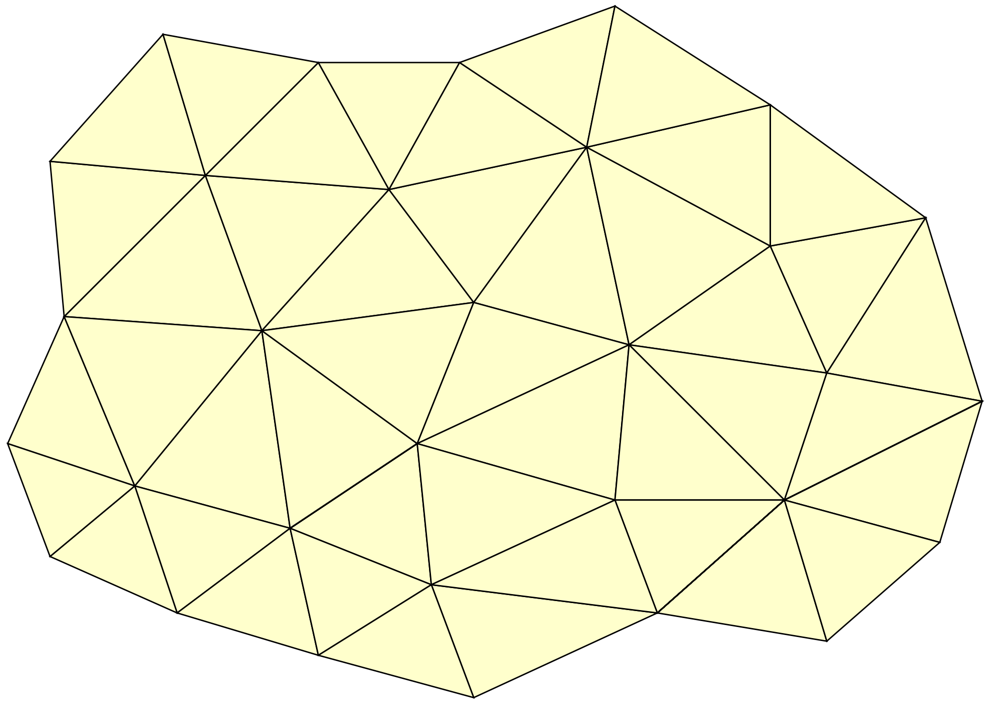
2D Quantum Gravity

Statistical Mechanics on a Regular Lattice



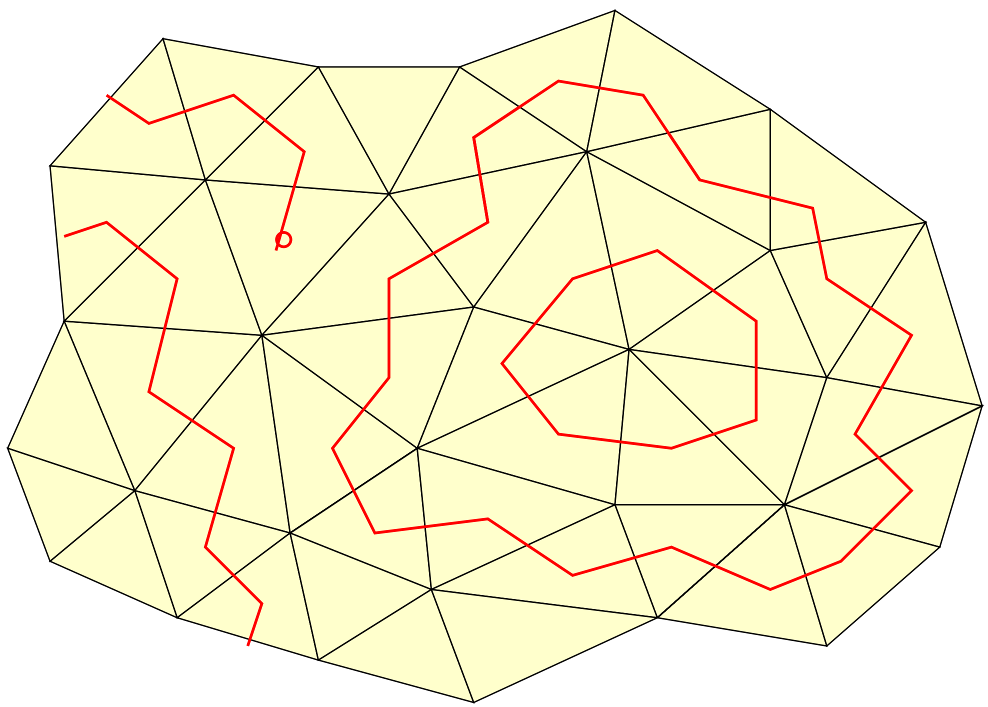
Random lines on the (dual of) a regular triangular lattice

Randomly Triangulated Lattice



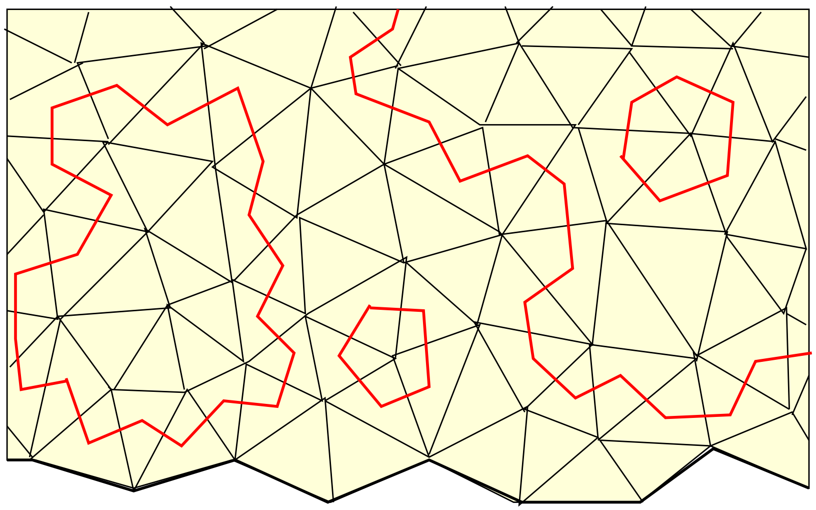
A random planar triangular lattice.

Statistical Mechanics on a Random Lattice



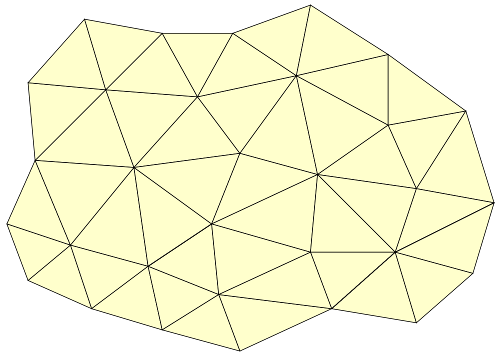
Statistical model on a random planar triangular lattice

Boundary Effects



Dirichlet boundary conditions on a random disk

Partition Function



Random triangular lattices G with fixed spherical topology.

$$Z(\beta) = \sum_{\text{planar } G} \frac{1}{S(G)} e^{-\beta|G|},$$

β: 'chemical potential' for the area, i.e., number of vertices $|G|$ of G; $S(G)$ its symmetry factor.

Thermodynamic Limit

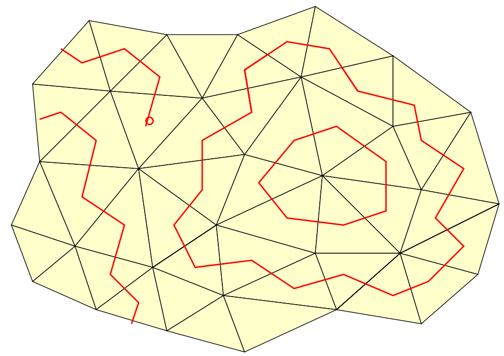
The partition sum converges for β larger than some critical β_c. At $\beta \to \beta_c^+$ a singularity appears due to infinite graphs

$$Z(\beta) \sim (\beta - \beta_c)^{2-\gamma_{\rm str}},$$

where $\gamma_{\rm str}$ is the string susceptibility exponent. For pure gravity and for the spherical topology

$$\gamma_{\rm str} = -\frac{1}{2}.$$

Partition Functions on a Random Lattice



Statistical model $\mathcal{M}$ on random lattice G

$$Z(\beta) = \sum_{\text{planar } G} \frac{1}{S(G)} e^{-\beta|G|} \mathcal{Z}_G$$

$\mathcal{Z}_G$: partition function of the statistical model $\mathcal{M}$ on G.

DOUBLE CRITICAL POINT of $\mathcal{M}$ & G

$$Z(\beta) \sim (\beta - \beta_c)^{2-\gamma_{\text{str}}(c)}$$

(c labels $\mathcal{M}$)

Double Critical Behavior

$\gamma_{\rm str}(c) \equiv \gamma$ is related to the "central charge" c of the CFT describing the statistical model by

$$c = 1 - 6\gamma^2/(1-\gamma)\,,\ \gamma \leqslant 0$$

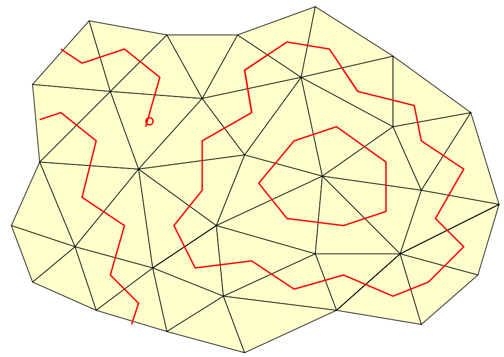
SLE_κ, $0 \leqslant \kappa \leqslant +\infty$

$$\gamma = 1 - \frac{4}{\kappa},\ \kappa \leqslant 4\,,\quad \gamma = 1 - \frac{\kappa}{4},\ 4 \leqslant \kappa$$

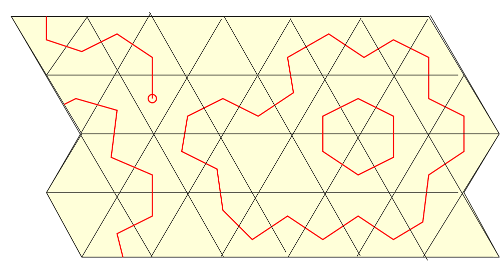
$$c = \frac{1}{4}(6-\kappa)\left(6 - \frac{16}{\kappa}\right)$$

Symmetric under *duality:* $\kappa \to \kappa' = 16/\kappa$

KPZ *Knizhnik, Polyakov, Zamolodchikov (88)*



A “conformal operator” O (e.g. creating the line extremity) has conformal weight Δ (or $\tilde{\Delta}$) in (boundary) quantum gravity.



The same operator has conformal weight $x = U(\Delta)$ in $\mathbb{C}$ (or $\tilde{x} = U(\tilde{\Delta})$ in $\mathbb{H}$).

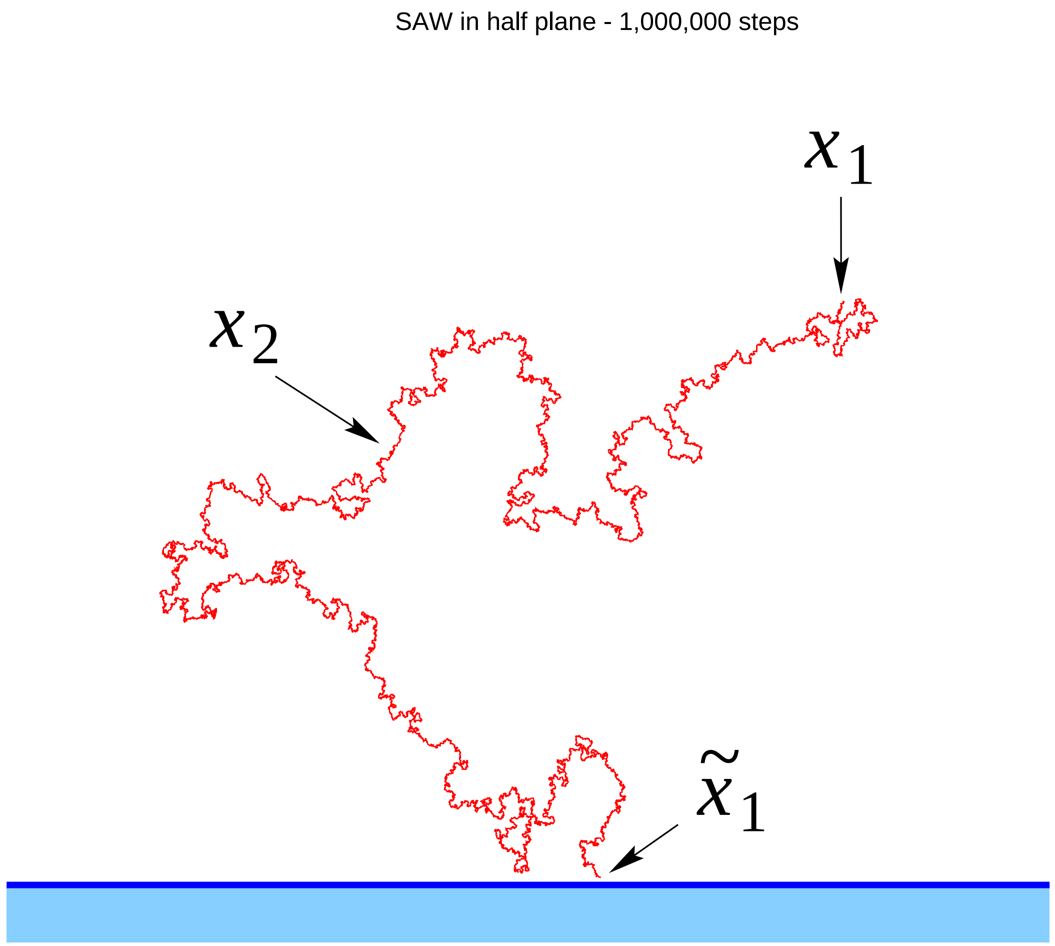
KPZ: A fundamental quadratic relation exists between the conformal dimensions Δ on a random planar surface and those x in $\mathbb{C}$

$$x = U(\Delta) = \Delta \frac{\Delta - \gamma}{1 - \gamma},$$

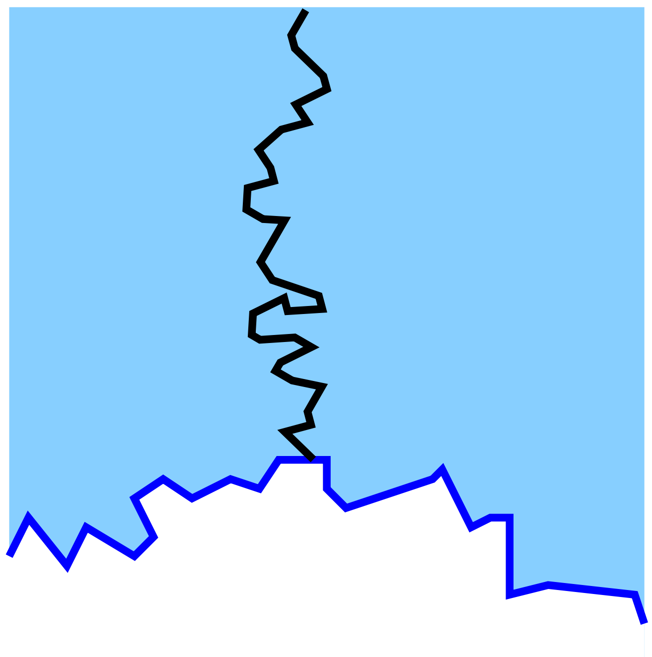
with the string susceptibility exponent γ related to the central charge c of the CFT

$$\begin{aligned} c &= 1 - 6\gamma^2/(1-\gamma), \qquad \gamma \leqslant 0, \quad c \leqslant 1 \\ &= \frac{1}{4}(6-\kappa)\left(6 - \frac{16}{\kappa}\right), \quad (\mathrm{SLE}_\kappa,\ 0 \leqslant \kappa \leqslant +\infty) \end{aligned}$$

Conformal Weights of a Path in $\mathbb{C}$ or $\mathbb{H}$



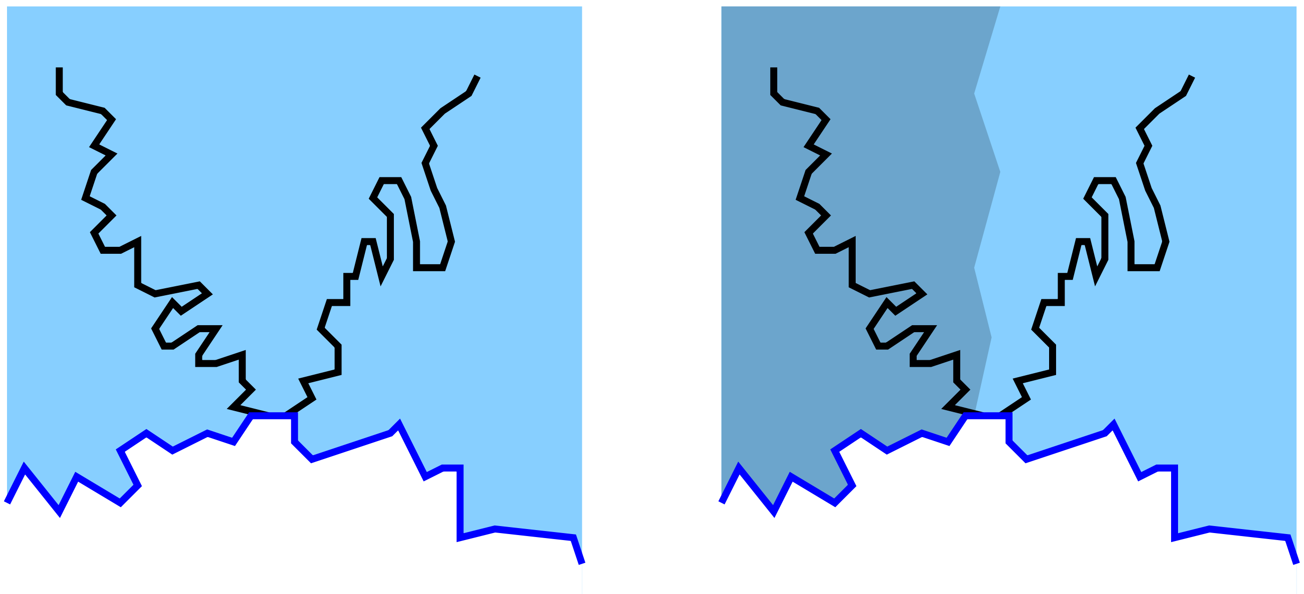
Boundary Conformal Weight in QG



$$\tilde{\Delta}_1 = U^{-1}(\tilde{x}_1)$$

$$= \frac{1-\gamma}{2}$$

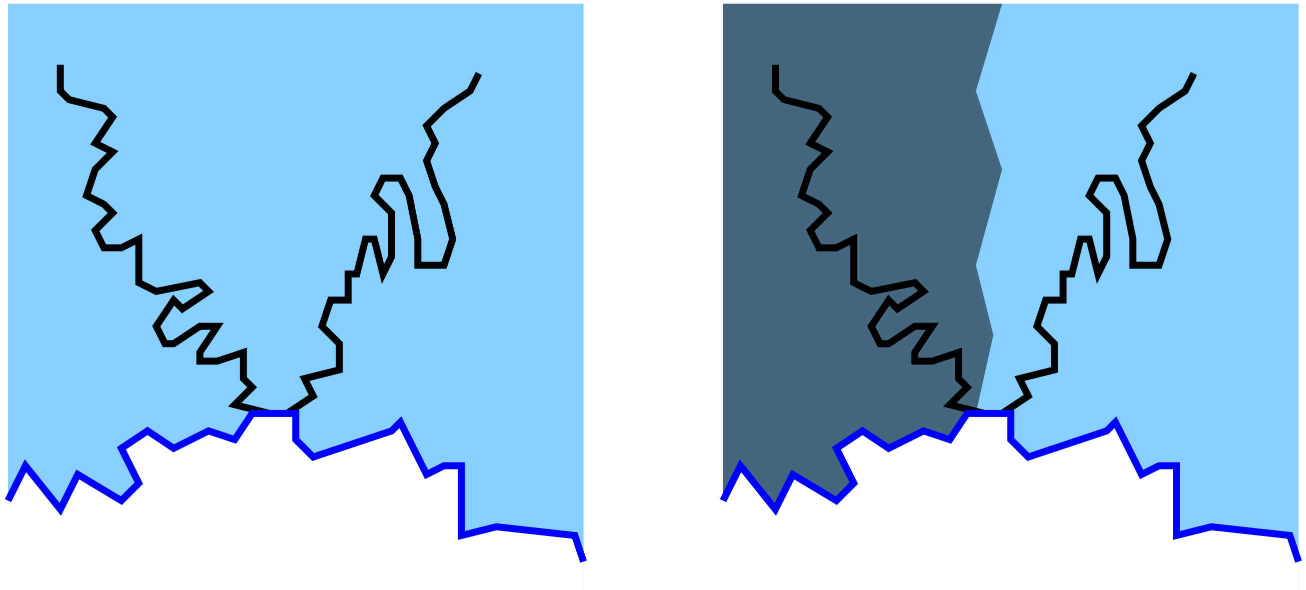
Boundary Quantum Gravity is Addi(c)tive



$$U^{-1}(\tilde{x}_2) = 2U^{-1}(\tilde{x}_1)$$

Boundary Quantum Gravity is

Additive for Mutual Avoidance

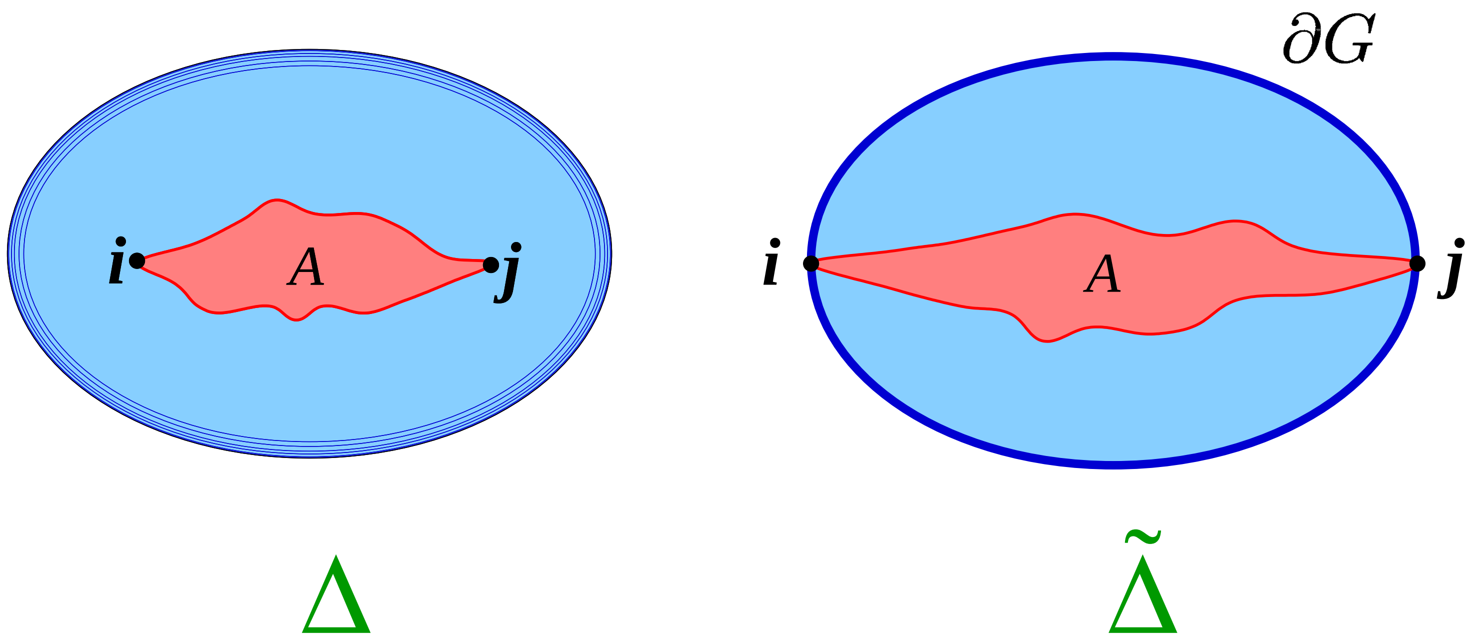


$$\tilde{\Delta}_2 = U^{-1}(\tilde{x}_2) = 2\,U^{-1}(\tilde{x}_1) = 2\,\tilde{\Delta}_1$$

$$= 1 - \gamma$$

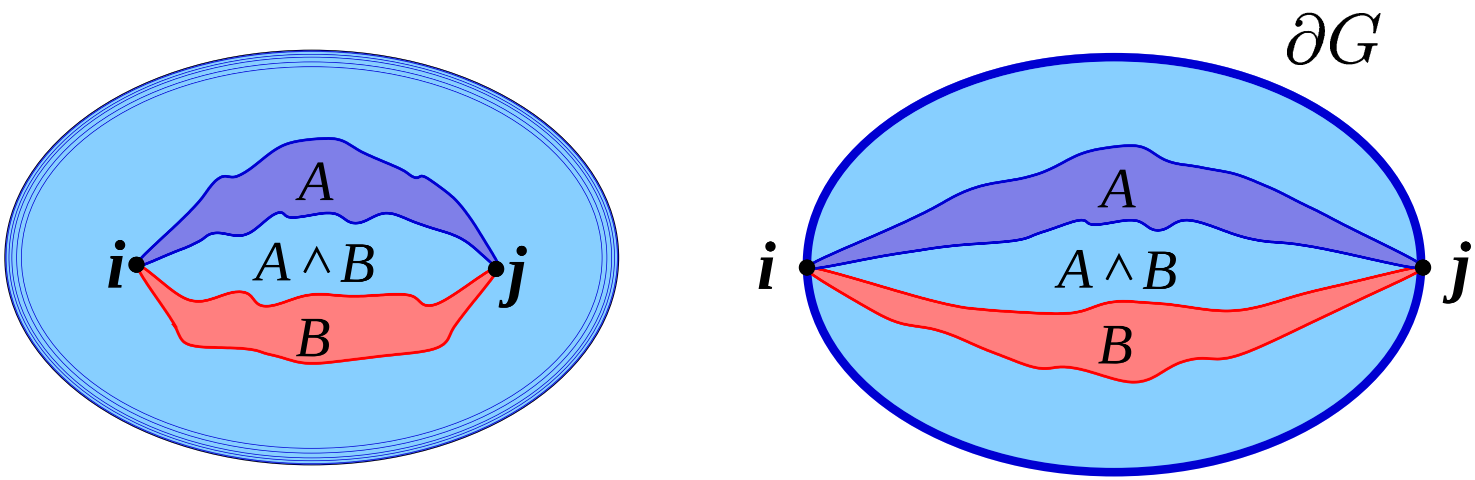
Life in QG is easy

Bulk-Boundary Relation



$$2\Delta - \gamma = \tilde{\Delta}$$

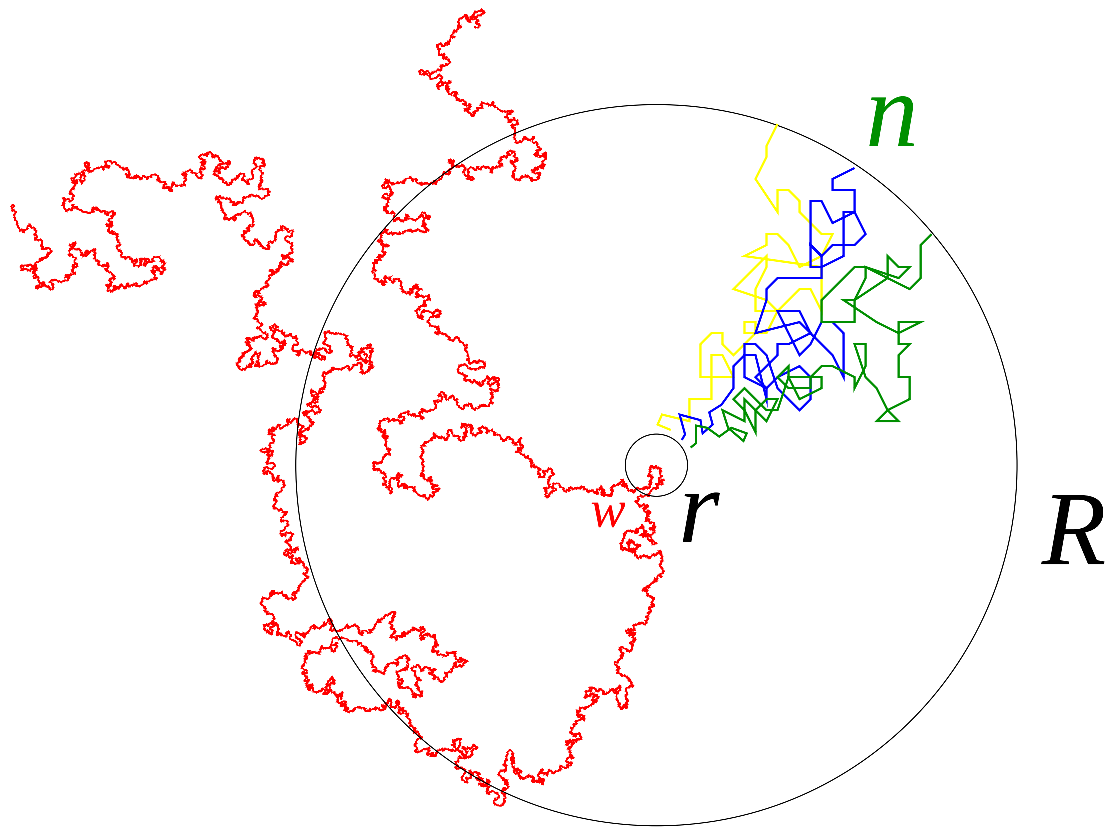
Quantum Boundary Additivity & Mutual Avoidance



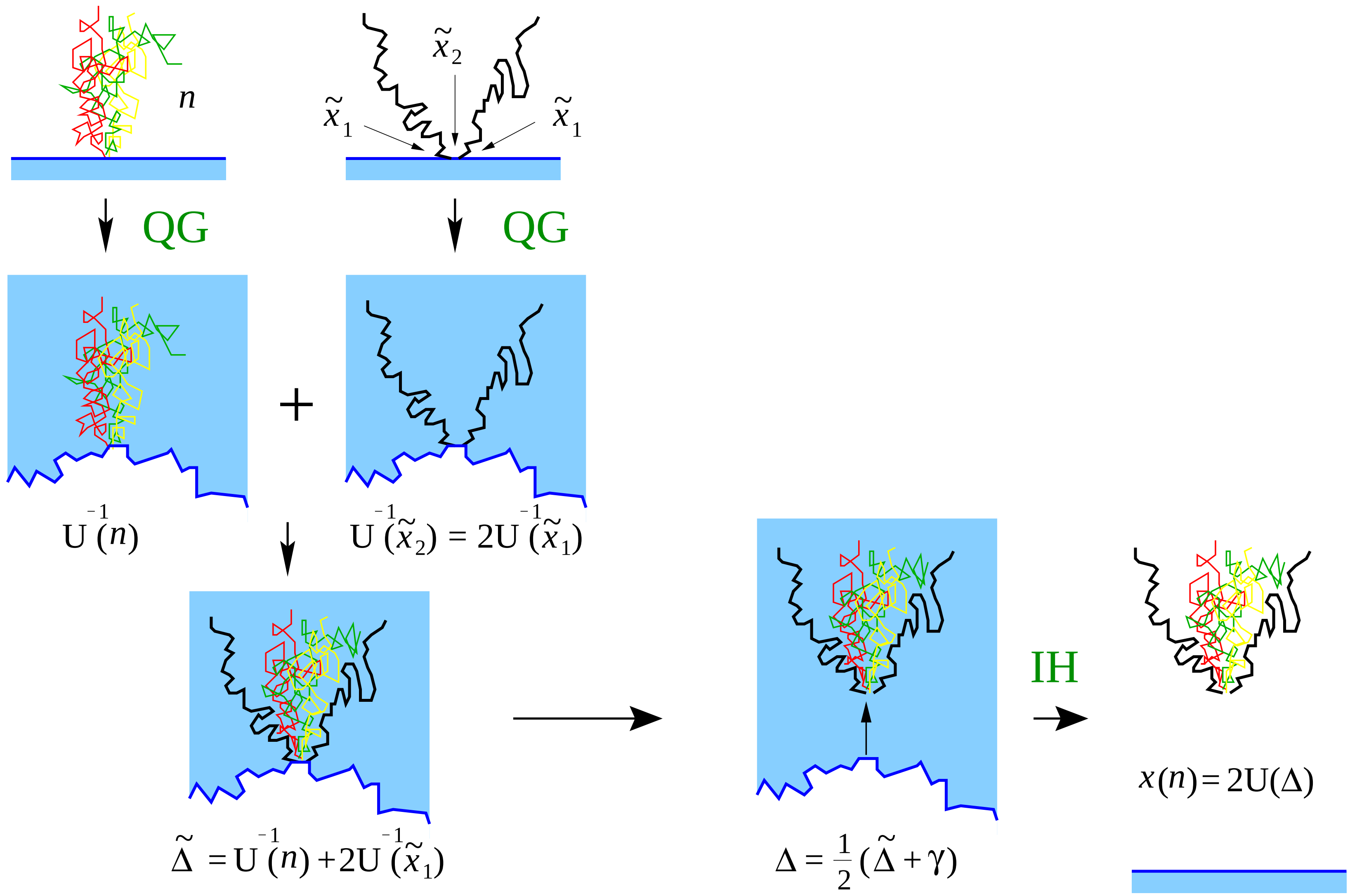
$$2\Delta_{A\wedge B} - \gamma = \tilde{\Delta}_{A\wedge B} = \tilde{\Delta}_A + \tilde{\Delta}_B$$

Multifractal Dimensions from QG

Multifractal Exponents $x(n)$



Quantum Gravity Construction



Quantum Gravity Construction

- Boundary

$$\tilde{\Delta} = U^{-1}(n) + 2U^{-1}(\tilde{\chi}_1)$$
$$= U^{-1}(n) + 1 - \gamma$$

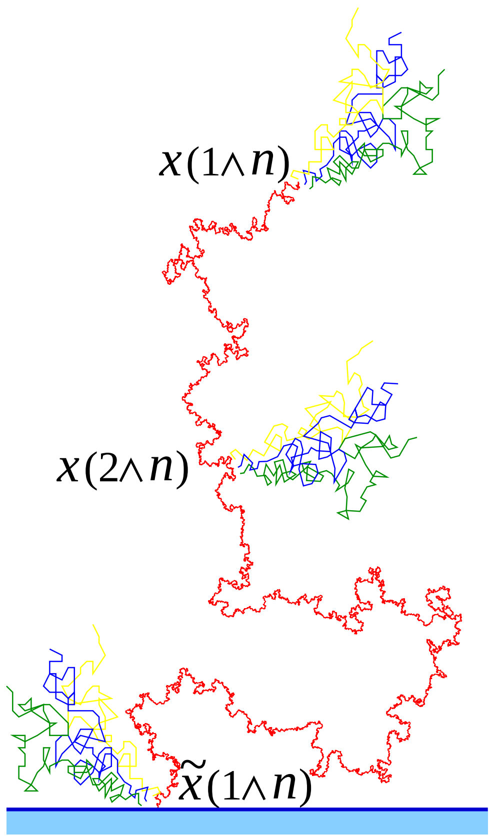
- Bulk

$$\Delta = \frac{1}{2}(\tilde{\Delta} + \gamma)$$
$$= \frac{1}{2}U^{-1}(n) + \frac{1}{2}$$

Multifractal Exponents

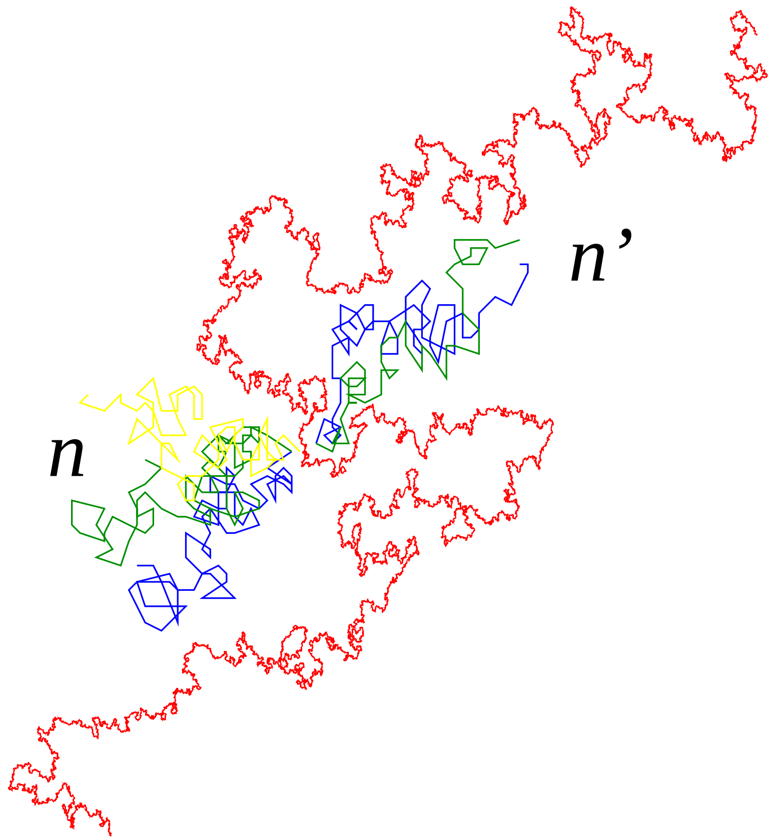
$$
\begin{aligned}
x(n) &= U(\Delta) \\
&= U\left(\frac{1}{2}U^{-1}(n) + \frac{1}{2}\right)
\end{aligned}
$$

Generalization I



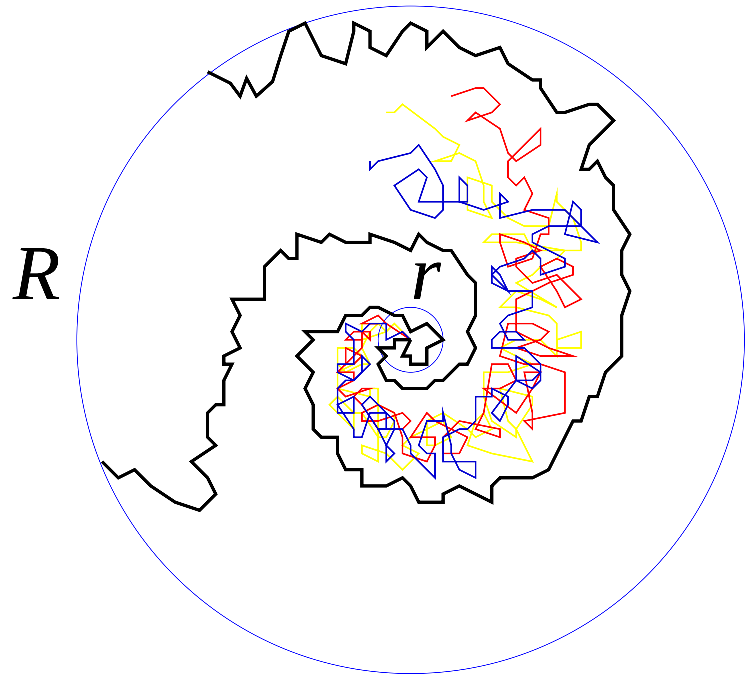
Packets of Brownian paths probing different locations (bulk tip, generic point, boundary tip) on the path frontier

Generalization II



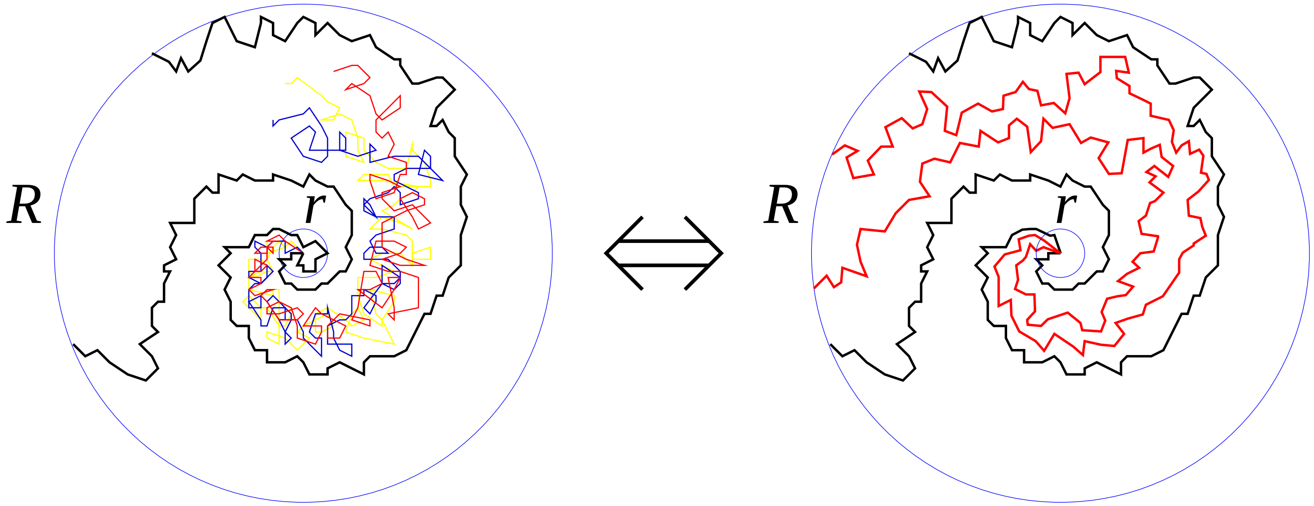
Double-sided moments for the double-sided potential

Rotation Exponents $x(n, p)$



A packet of n independent random walks winding with & avoiding the two frontier paths.

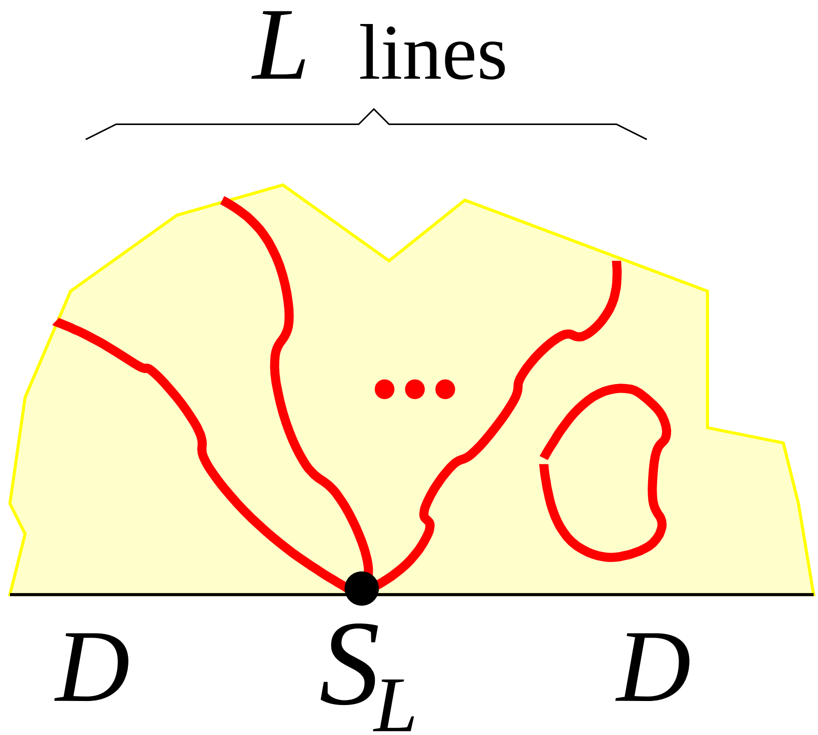
Path Equivalence



Total Number of Simple Paths: $L = 2 + \#$

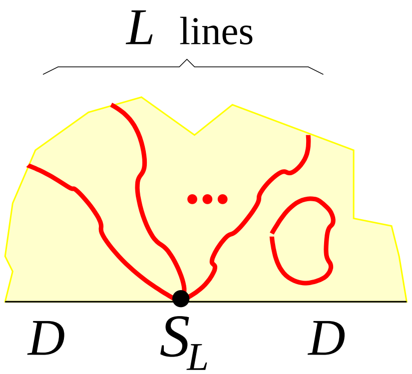
n independent Brownian paths $\Longleftrightarrow$ # mutually-avoiding simple paths,

$\# = \frac{U^{-1}(n)}{U^{-1}(\tilde{x}_1)}$ as determined from QG



A S_L *star vertex at the* Dirichlet *boundary, with conformal weight* $\tilde{\Delta}_L$

Number of Equivalent Paths

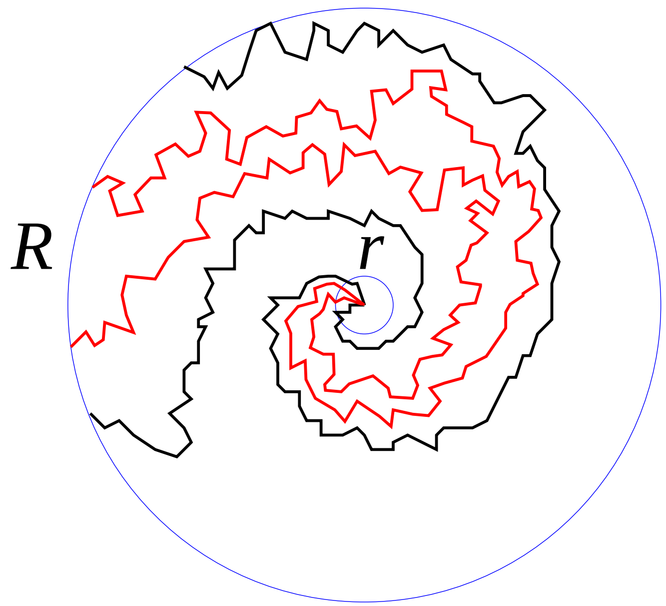


$$\tilde{\Delta}_L = L \times \tilde{\Delta}_1 = L \times U^{-1}(\tilde{x}_1)$$

$$\tilde{\Delta} = 2U^{-1}(\tilde{x}_1) + U^{-1}(n)$$

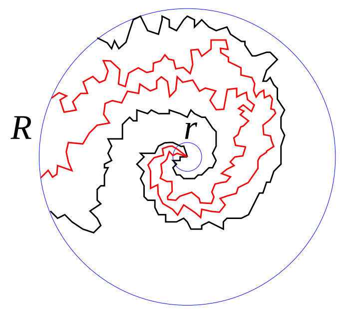
$$\tilde{\Delta}_L = \tilde{\Delta} \iff L = 2 + \frac{U^{-1}(n)}{U^{-1}(\tilde{x}_1)}$$

Coulomb Gas & Windings



$$x(n,p) = x(n) - \frac{1}{1-\gamma}\frac{p^2}{L^2}$$

Equivalent Path #: $L = 2 + \frac{U^{-1}(n)}{U^{-1}(\tilde{x}_1)}$



$$x(n,p) = x(n) - \frac{1}{8}\frac{p^2}{2x(n)+b-2},$$

$$b = \frac{25-c}{12}$$

Multifractal Scaling Law

By Double Legendre Transform of $x(n, p)$

$$f(\alpha, \lambda) = (1+\lambda^2) f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2$$

$$b = \frac{25-c}{12}$$

Multifractal Spectra

Legendre Transform & Scaling Law

$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}, \qquad b = \frac{25 - c}{12}$$

$$f(\alpha, \lambda) = (1+\lambda^2) f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2$$

$$= \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2}$$

(B.D., 1999; B.D. & I. Binder, 2002)

Universal Multifractal Exponents

$$\tau(n) = 2x(n) - 2$$
$$= \frac{1}{2}(n-1) + \frac{25-c}{24}\left(\sqrt{\frac{24n+1-c}{25-c}} - 1\right)$$

Legendre Transform

$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}$$
$$b = \frac{25-c}{12}$$

Mixed Multifractal Spectrum

$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}$$

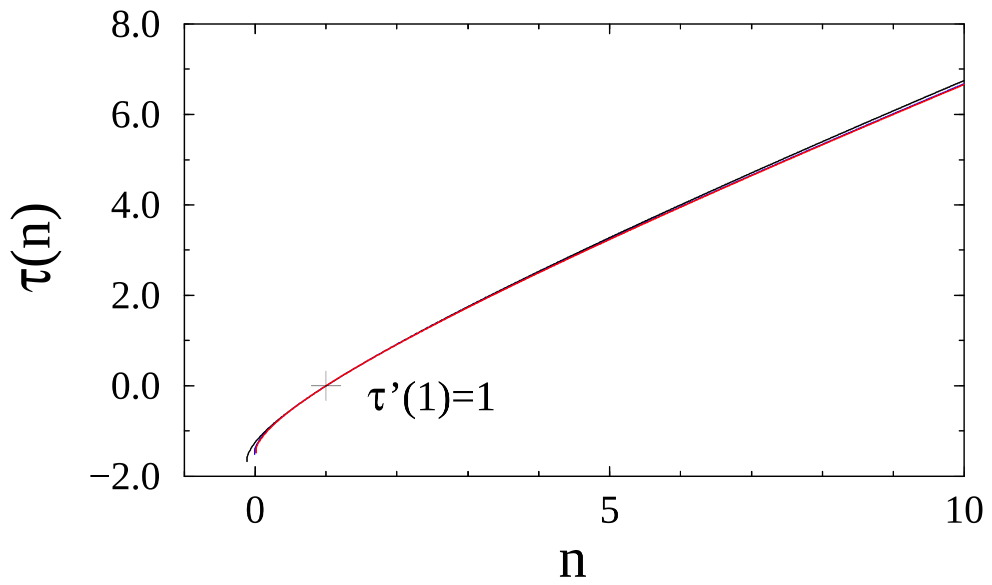
$$\begin{aligned} f(\alpha, \lambda) &= (1+\lambda^2) f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2 \\ &= \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2} \\ b &= \frac{25 - c}{12}. \end{aligned}$$

Summary: Multifractal Functions

$$\tau(n) = \frac{1}{2}(n-1) + \frac{25-c}{24}\left(\sqrt{\frac{24n+1-c}{25-c}} - 1\right)$$

$$\begin{aligned} f(\alpha) &= \alpha + b - \frac{b\alpha^2}{2\alpha - 1} \\ f(\alpha,\lambda) &= \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2} \\ b &= \frac{25-c}{12}. \end{aligned}$$

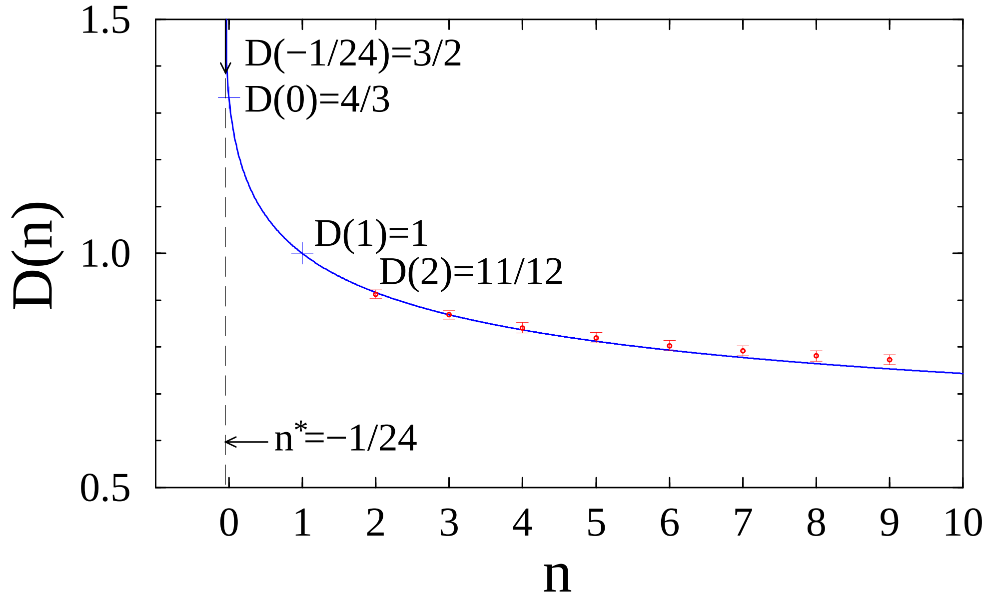
Universal Exponents $\tau(n) = 2x(n) - 2$



$$\tau(n) = \frac{1}{2}(n-1) + \frac{25-c}{24}\left(\sqrt{\frac{24n+1-c}{25-c}} - 1\right)$$

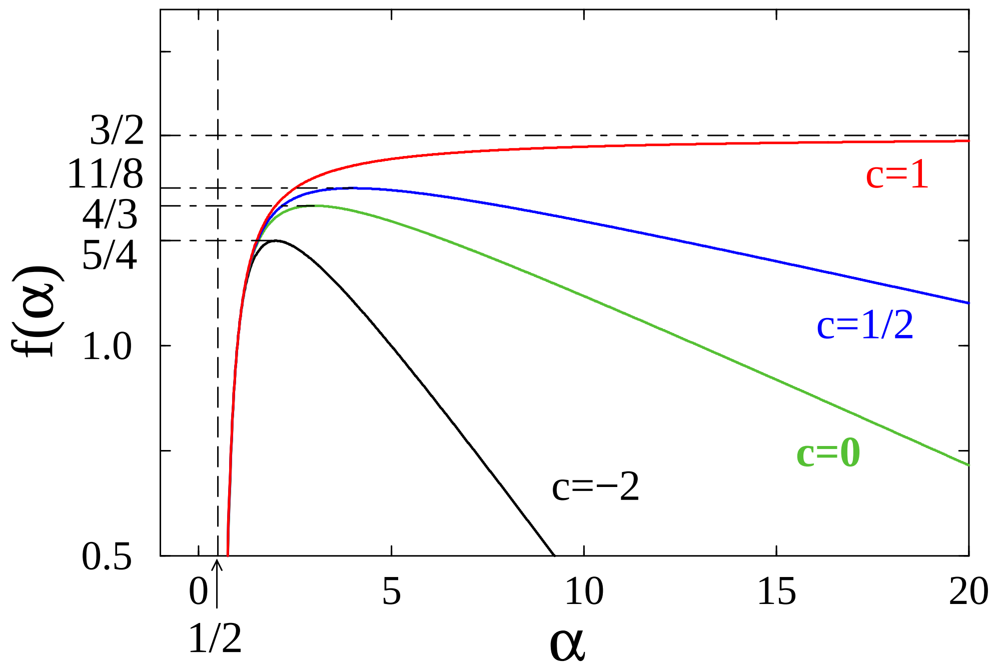
$$c = -2, 0, 1/2, 1$$

Generalized Dimensions



$D(n) = \frac{\tau(n)-1}{n-1}$ (blue line), corresponding to a percolation cluster, a self-avoiding or a random walk; comparison with the numerical data (red points) obtained by Meakin et al. (1988) for percolation

Multifractal Spectra $f(\alpha)$



Loop-erased RW ($c = -2$, SLE_2); Brownian & percolation frontiers, and SAW's ($c = 0$, $\mathrm{SLE}_{8/3}$); Ising clusters ($c = \frac{1}{2}$, SLE_3); $Q = 4$ Potts clusters ($c = 1$, SLE_4).

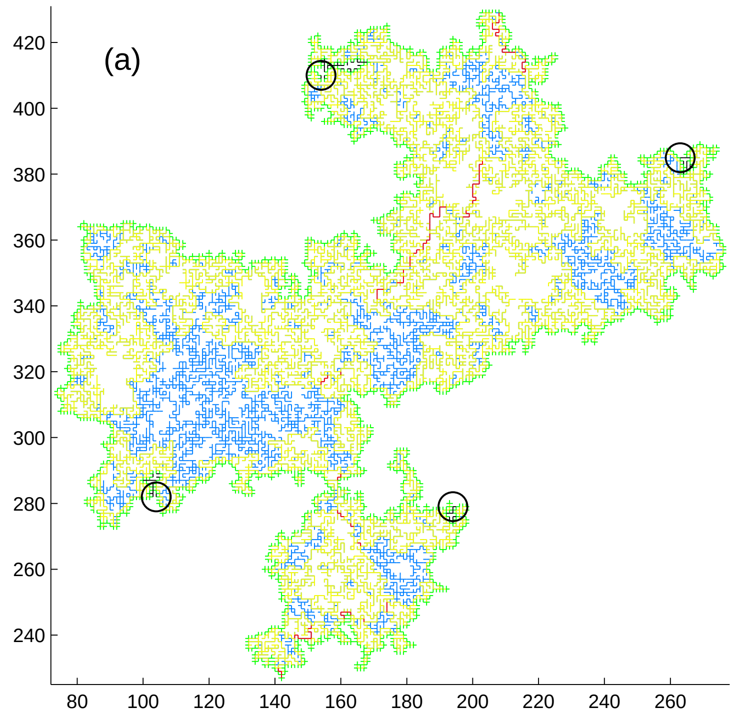
External Perimeter Dimension

$$D_{\mathrm{EP}} = \sup_{\alpha,\lambda} f(\alpha,\lambda) = \sup_{\alpha} f(\alpha,\lambda=0) = \sup_{\alpha} f(\alpha)$$

$$= \frac{3}{2} - \frac{1}{24}\sqrt{1-c}\left(\sqrt{25-c}-\sqrt{1-c}\right)$$

$$c \leqslant 1$$

Percolation Hull & Frontier



Cluster; Hull: $D_{\mathrm{Hull}} = \frac{7}{4}$ *(DS, 1987; Smirnov; Beffara); External Perimeter:* $D_{\mathrm{EP}} = \frac{4}{3}$ *(ADA, 1999); (courtesy of J. Asikainen, et al., 2003).*

Duality

Hull & External Perimeter Dimensions

$$D_{\mathrm{Hull}} = \frac{7}{4} \geqslant \frac{3}{2}$$

$$\left(D_{\mathrm{Hull}} - 1\right)\left(D_{\mathrm{EP}} - 1\right) = \frac{1}{4}$$

$$D_{\mathrm{EP}} = \frac{4}{3} \leqslant \frac{3}{2}$$

SLE External Perimeter & Hull

$$c = \frac{1}{4}(6-\kappa)\left(6-\frac{16}{\kappa}\right)$$

$$D_{\mathrm{EP}} = 1+\frac{\kappa}{8}\vartheta(4-\kappa)+\frac{2}{\kappa}\vartheta(\kappa-4)$$

$$D_{\mathrm{Hull}} = 1+\frac{\kappa}{8}$$

SLE Duality

$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa), \ \kappa \leqslant 4$$

$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa' = 16/\kappa), \quad \kappa \geqslant 4$$

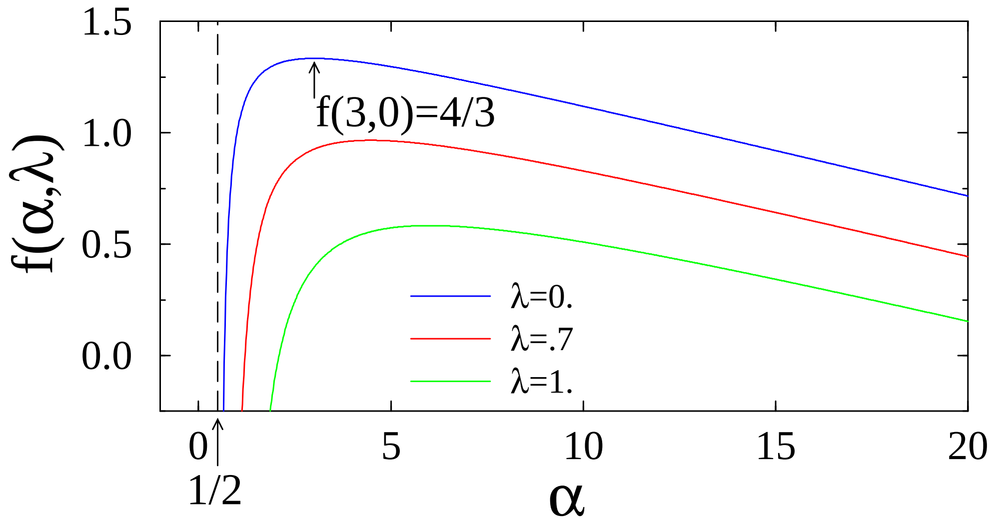
$$\frac{1}{4} = [D_{\mathrm{EP}}(\kappa) - 1][D_{\mathrm{H}}(\kappa) - 1]$$

Duality: the external perimeter of $\mathrm{SLE}_{\kappa \geqslant 4}$ is the simple path of $\mathrm{SLE}_{[(16/\kappa) \leqslant 4]}$

S	$D_S(Q=1)$		$D_S(Q=2)$		$D_S(Q=3)$		$D_S(Q=4)$	
	n	e	n	e	n	e	n	e
M	$1.90(1)$	$\frac{91}{48}$	$1.87(1)$	$\frac{15}{8}$	$1.85(2)$	$\frac{28}{15}$	$2.05(15)$	$\frac{15}{8}$
H	$1.75(1)$	$\frac{7}{4}$	$1.66(1)$	$\frac{5}{3}$	$1.59(3)$	$\frac{8}{5}$	$1.50(1)$	$\frac{3}{2}$
EP	$1.33(5)$	$\frac{4}{3}$	$1.36(2)$	$\frac{11}{8}$	$1.40(2)$	$\frac{17}{12}$	$1.48(2)$	$\frac{3}{2}$
SC	$0.75(2)$	$\frac{3}{4}$	$0.55(3)$	$\frac{13}{24}$	$0.35(7)$	$\frac{7}{20}$	$0.03(8)$	0
G	$-0.90(5)$	$-\frac{11}{12}$	$-0.71(5)$	-	$-0.63(5)$	-	$-0.59(5)$	-

Table 1: *Comparison of the numerical estimates (n) (Asikainen et al. 2003) for the subset fractal dimensions D_S with the exact predictions (e) where available. M: cluster mass; H: Hull; EP: External Perimeter; SC: Singly Connected Bonds; G: gates.*

Mixed Spectra

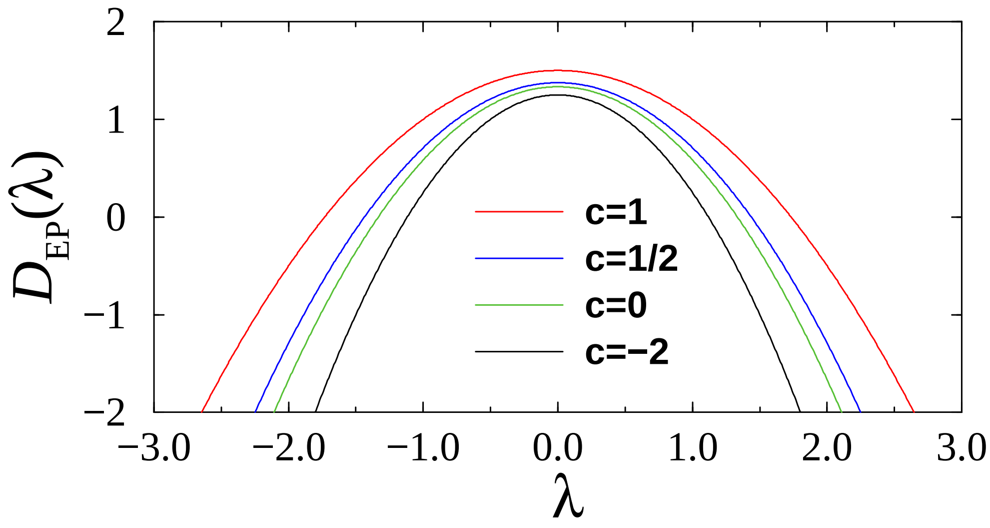


Mixed multifractal spectrum $f(\alpha,\lambda)$ *for* $c=0$ *(Brownian frontier, percolation EP and SAW), and for various spiralling rates* λ*. The maximum* $f(\alpha=3,\lambda=0)=4/3$ *is the Hausdorff dimension of the frontier.*

Rotation Dimensions

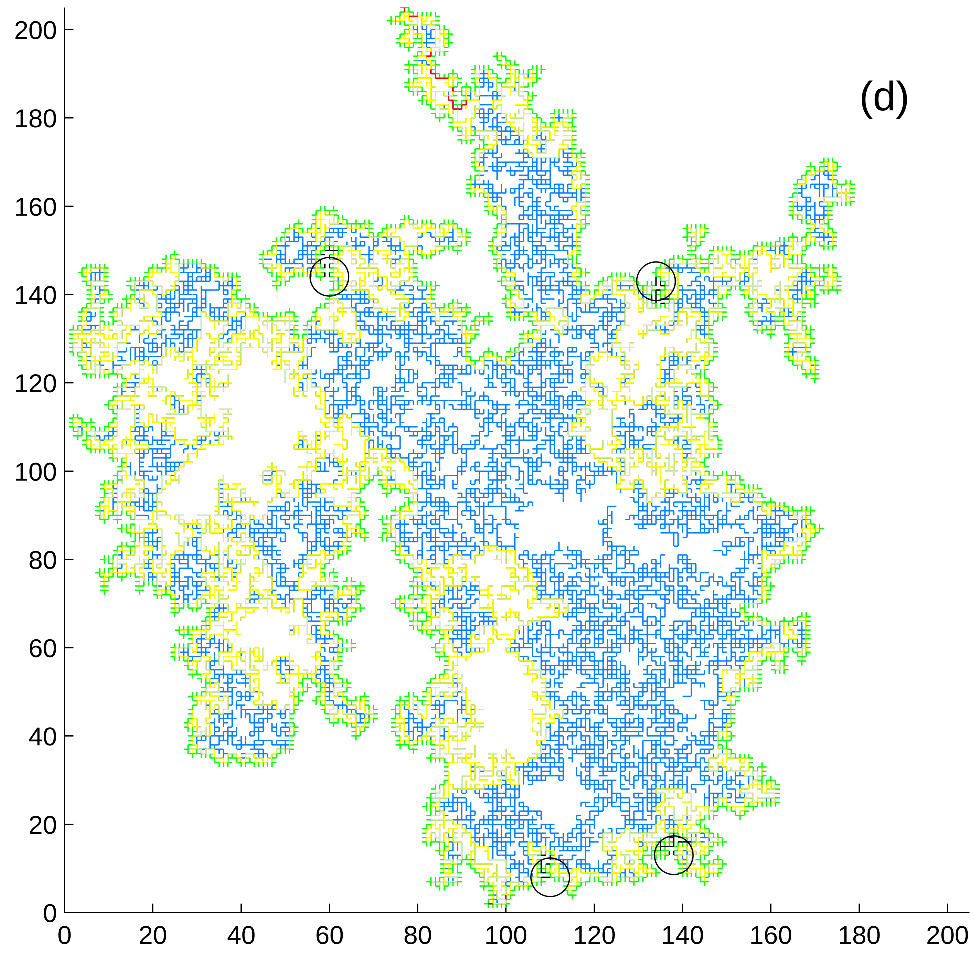
$$D_{\mathrm{EP}}(\lambda) = \sup_{\alpha} f(\alpha, \lambda)$$
$$= D_{\mathrm{EP}} - (b - D_{\mathrm{EP}})\,\lambda^2$$

$$b \geqslant 2, \quad D_{\mathrm{EP}} \leqslant \frac{3}{2}$$



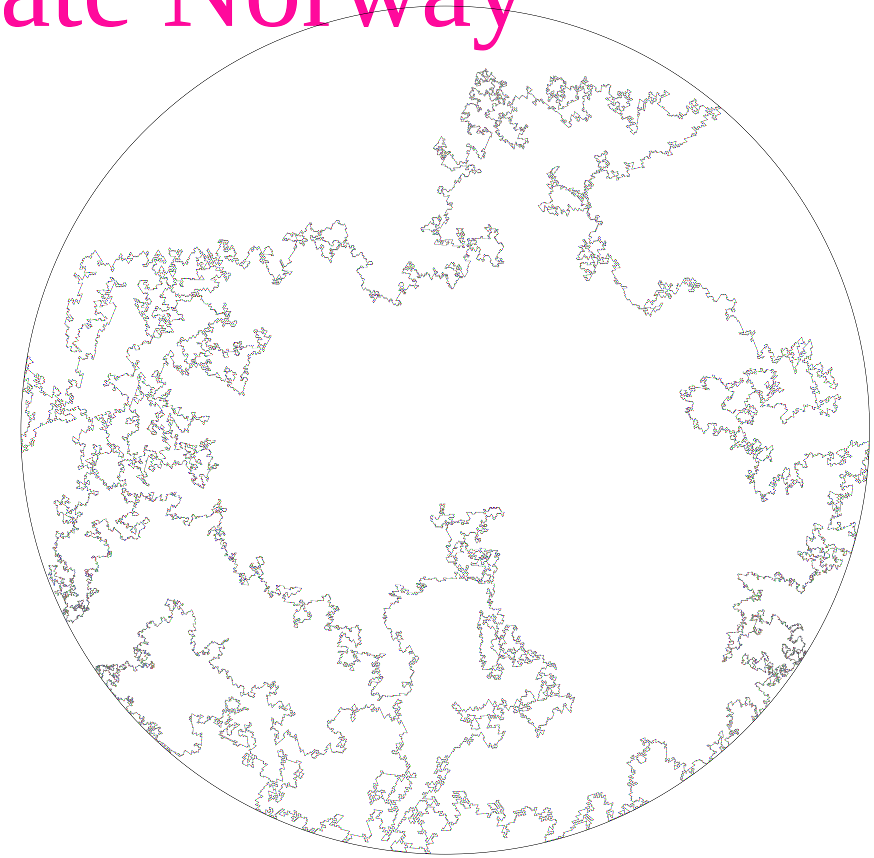
Dimensions $D_{\mathrm{EP}}(\lambda)$ of the external frontier as a function of rotation rate λ: loop-erased RW ($c = -2; \mathrm{SLE}_2$); Brownian & percolation frontiers, and SAW's ($c = 0; \mathrm{SLE}_{8/3}$); Ising clusters ($c = \frac{1}{2}; \mathrm{SLE}_3$); $Q = 4$ Potts clusters ($c = 1; \mathrm{SLE}_4$) (or "Ultimate Norway").

Potts FK Cluster $(Q = 4)$



Cluster; *Hull*; *External Perimeter*.

Ultimate Norway



The "Ultimate Norway", i.e. the frontier of a $Q = 4$ *Potts cluster or* $\mathrm{SLE}_{\kappa=4}$, *the self-dual conformally invariant random curve* $(c = 1)$ *with maximal Hausdorff dimension* $D = 3/2$ *(courtesy of D. Wilson).*