# Some wonderful conjectures (but very few theorems) at the boundary between analysis, combinatorics and probability

Alan Sokal
New York University / University College London

Mark Kac Seminar, Utrecht
March 7, 2014

**References:**

1. Roots of a formal power series $f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$, with applications to graph enumeration and $q$-series,
Series of 4 lectures at Queen Mary (London), March–April 2011,
`http://www.maths.qmw.ac.uk/~pjc/csgnotes/sokal/`

2. The leading root of the partial theta function, arXiv:1106.1003 [math.CO], Adv. Math. **229**, 2603–2621 (2012).

The deformed exponential function $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$

- Defined for complex $x$ and $y$ satisfying $|y| \le 1$
- Analytic in $\mathbb{C} \times \mathbb{D}$, continuous in $\mathbb{C} \times \overline{\mathbb{D}}$
- $F(\,\cdot\,, y)$ is entire for each $y \in \overline{\mathbb{D}}$
- Valiron (1938): "from a certain viewpoint the simplest entire function after the exponential function"

**Applications:**

- Statistical mechanics: Partition function of one-site lattice gas
- Combinatorics: Enumeration of connected graphs,
  generating function for Tutte polynomials on $K_n$
  (also acyclic digraphs, inversions of trees, ...)
- Functional-differential equation: $F'(x) = F(yx)$ where $' = \partial/\partial x$
- Complex analysis: Whittaker and Goncharov constants

## Application to enumeration of connected graphs

- Let $a_{n,m} = \#$ graphs with $n$ labelled vertices and $m$ edges
- Generating polynomial $A_n(v) = \sum\limits_m a_{n,m}\, v^m$
- Exponential generating function $A(x,v) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} A_n(v)$
- Of course $a_{n,m} = \binom{n(n-1)/2}{m} \quad \Longrightarrow \quad A_n(v) = (1+v)^{n(n-1)/2} \quad \Longrightarrow$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} A_n(v) \;=\; F(x, 1+v)$$

- Now let $c_{n,m} = \#$ *connected* graphs with $n$ labelled vertices and $m$ edges
- Generating polynomial $C_n(v) = \sum\limits_m c_{n,m}\, v^m$
- Exponential generating function $C(x,v) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n!} C_n(v)$
- No simple explicit formula for $C_n(v)$ is known, but ...
- The *exponential formula* tells us that $C(x,v) = \log A(x,v)$, i.e.

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) \;=\; \log F(x, 1+v)$$

  [see Tutte (1967) and Scott–A.D.S., arXiv:0803.1477 for generalizations to the Tutte polynomials of the complete graphs $K_n$]

- Usually considered as formal power series
- But series are *convergent* if $|1+v| \le 1$
  [see also Flajolet–Salvy–Schaeffer (2004)]

Elementary analytic properties of $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$

- $\boldsymbol{y = 0}$: $F(x,0) = 1 + x$

- $\boldsymbol{0 < |y| < 1}$: $F(\,\cdot\,, y)$ is a nonpolynomial entire function of order 0:
$$F(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right)$$
where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 0$

- $\boldsymbol{y = 1}$: $F(x,1) = e^x$

- $\boldsymbol{|y| = 1}$ **with** $\boldsymbol{y \neq 1}$: $F(\,\cdot\,, y)$ is an entire function of order 1 and type 1:
$$F(x,y) \;=\; e^x \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right) e^{x/x_k(y)} \;.$$
where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 1$

  [see also Ålander (1914) for $y$ a root of unity; Valiron (1938) and Eremenko–Ostrovskii (2007) for $y$ not a root of unity]

- $\boldsymbol{|y| > 1}$: The series $F(\,\cdot\,, y)$ has radius of convergence 0

## Consequences for $C_n(v)$

- Make change of variables $y = 1 + v$:

$$\overline{C}_n(y) \;=\; C_n(y-1)$$

- Then for $|y| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \overline{C}_n(y) \;=\; \log F(x,y) \;=\; \sum_k \log\Bigl(1 \,-\, \frac{x}{x_k(y)}\Bigr)$$

  and hence

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n} \qquad \text{for all } n \ge 1$$

  (also holds for $n \ge 2$ when $|y| = 1$)

- This is a *convergent* expansion for $\overline{C}_n(y)$
- In particular, gives large-$n$ asymptotic behavior

$$\overline{C}_n(y) \;=\; -(n-1)!\, x_0(y)^{-n} \left[1 \,+\, O(e^{-\epsilon n})\right]$$

  whenever $F(\,\cdot\,, y)$ has a unique root $x_0(y)$ of minimum modulus

**Question:** What can we say about the roots $x_k(y)$?

## Small-$y$ expansion of roots $x_k(y)$

- For small $|y|$, we have $F(x, y) = 1 + x + O(y)$, so we expect a convergent expansion

$$x_0(y) \;=\; -1 \;-\; \sum_{n=1}^{\infty} a_n y^n$$

(easy proof using Rouché: valid for $|y| \lesssim 0.441755$)

- More generally, for each integer $k \geq 0$, write $x = \xi y^{-k}$ and study

$$F_k(\xi, y) \;=\; y^{k(k+1)/2} F(\xi y^{-k}, y) \;=\; \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, y^{(n-k)(n-k-1)/2}$$

Sum is dominated by terms $n = k$ and $n = k + 1$; gives root

$$x_k(y) \;=\; -(k+1)y^{-k} \left[ 1 + \sum_{n=1}^{\infty} a_n^{(k)} y^n \right]$$

Rouché argument valid for $|y| \lesssim 0.207875$ uniformly in $k$:
all roots are simple and given by convergent expansion $x_k(y)$

- Can also use theta function in Rouché (Eremenko)

Might these series converge for all $|y| < 1$?

Two ways that $x_k(y)$ could fail to be analytic for $|y| < 1$:

1. Collision of roots ($\to$ branch point)
2. Root escaping to infinity

**Theorem (Eremenko):** No root can escape to infinity for $y$ in the open unit disc $\mathbb{D}$ (except of course at $y = 0$).

In fact, for any compact subset $K \subset \mathbb{D}$ and any $\epsilon > 0$, there exists an integer $k_0$ such that for all $y \in K \setminus \{0\}$ we have:

(a) The function $F(\,\cdot\,, y)$ has exactly $k_0$ zeros (counting multiplicity) in the disc $|x| < k_0 |y|^{-(k_0 - \frac{1}{2})}$, and

(b) In the region $|x| \geq k_0 |y|^{-(k_0 - \frac{1}{2})}$, the function $F(\,\cdot\,, y)$ has a simple zero within a factor $1+\epsilon$ of $-(k+1)y^{-k}$ for each $k \geq k_0$, and no other zeros.

- Proof is based on comparison with a theta function (whose roots are known by virtue of Jacobi's product formula)
- *Conjecture* that roots cannot escape to infinity even in the *closed* unit disc except at $y = 1$

**Big Conjecture #1.** All roots of $F(\,\cdot\,, y)$ are simple for $|y| < 1$. [and also for $|y| = 1$, I suspect]

**Consequence of Big Conjecture #1.** Each root $x_k(y)$ is analytic in $|y| < 1$.

## But I conjecture more ...

**Big Conjecture #2.** The roots of $F(\,\cdot\,, y)$ are non-crossing *in modulus* for $|y| < 1$:

$$|x_0(y)| \;<\; |x_1(y)| \;<\; |x_2(y)| \;<\; \ldots$$

[and also for $|y| = 1$, I suspect]

**Consequence of Big Conjecture #2.** The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| \;<\; |y|\,|x_{k+1}(y)| \qquad \text{for all } k \geq 0$$

Proof. Apply the Schwarz lemma to $x_k(y)/x_{k+1}(y)$.

## Consequence for the zeros of $\overline{C}_n(y)$

Recall

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$$

and use a variant of the Beraha–Kahane–Weiss theorem [A.D.S., arXiv:cond-mat/0012369, Theorem 3.2] $\Longrightarrow$ the limit points of zeros of $\overline{C}_n$ are the values $y$ for which the zero of minimum modulus of $F(\,\cdot\,, y)$ is *nonunique*.

So if $F(\,\cdot\,, y)$ has a *unique* zero of minimum modulus for all $y \in \mathbb{D}$ (a weakened form of Big Conjecture #2), then the zeros of $\overline{C}_n$ do not accumulate anywhere in the open unit disc.

I actually conjecture more (based on computations up to $n \approx 80$):

**Big Conjecture #3.** For each $n$, $\overline{C}_n(y)$ has no zeros with $|y| < 1$. [and, I suspect, no zeros with $|y| = 1$ except the point $y = 1$]

What is the evidence for these conjectures?

**Evidence #1:** Behavior at real $y$.

**Theorem (Laguerre):** For $0 \le y < 1$, all the roots of $F(\,\cdot\,, y)$ are simple and negative real.

**Corollary:** Each root $x_k(y)$ is analytic in a complex neighborhood of the interval $[0, 1)$.

[Real-variables methods give further information about the roots $x_k(y)$ for $0 \le y < 1$: see Langley (2000).]

Now combine this with

**Evidence #2:** From numerical computation of the series $x_k(y)$ ... [algorithms to be discussed later]

Let MATHEMATICA run for a weekend . . .

$$
\begin{aligned}
-x_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\
& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
& + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\
& + \ldots + \text{terms through order } y^{899}
\end{aligned}
$$

and all the coefficients (so far) are nonnegative!

- Very recently I have computed $x_0(y)$ through order $y^{16383}$.
- I also have shorter series for $x_k(y)$ for $k \geq 1$.

**Big Conjecture #4.** For each $k$, the series $-x_k(y)$ has all nonnegative coefficients.

Combine this with the known analyticity for $0 \leq y < 1$, and Pringsheim gives:

**Consequence of Big Conjecture #4.** Each root $x_k(y)$ is analytic in the open unit disc.

**NEED TO DO:** Extended computations for $k = 1, 2, \ldots$ and for symbolic $k$.

But more is true …

Look at the *reciprocal* of $x_0(y)$:

$$
\begin{aligned}
-\frac{1}{x_0(y)} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
& - \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\
& - \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\
& - \ldots - \text{terms through order } y^{899}
\end{aligned}
$$

and all the coefficients (so far) beyond the constant term are *nonpositive*!

**Big Conjecture #5.** For each $k$, the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

[This implies the preceding conjecture, but is stronger.]

- Relative simplicity of the coefficients of $-1/x_0(y)$ compared to those of $-x_0(y)$ $\longrightarrow$ simpler combinatorial interpretation?
- Note that $x_k(y) \to -\infty$ as $y \uparrow 1$ (this is fairly easy to prove). So $1/x_k(y) \to 0$. Therefore:

**Consequence of Big Conjecture #5.** For each $k$, the coefficients (after the constant term) in the series $-(k+1)y^{-k}/x_k(y)$ are the *probabilities* for a positive-integer-valued random variable.

What might such a random variable be???
Could this approach be used to *prove* Big Conjecture #5?
(see also the next two slides)

But I conjecture that even more is true ...

Define $D_n(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ and recall that $-x_0(y) = \lim_{n\to\infty} D_n(y)^{-1/n}$

**Big Conjecture #6.** For each $n$,

(a) the series $D_n(y)^{-1/n}$ has all nonnegative coefficients,

and even more strongly,

(b) the series $D_n(y)^{1/n}$ has all nonpositive coefficients after the constant term 1.

Since $D_n(y) > 0$ for $0 \le y < 1$, Pringsheim shows that Big Conjecture #6a implies Big Conjecture #3:

For each $n$, $\overline{C}_n(y)$ has no zeros with $|y| < 1$.

Moreover, Big Conjecture #6b $\implies$ for each $n$, the coefficients in the series $1 - D_n(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.

Such a random variable would generalize the one for $-1/x_0(y)$ in roughly the same way that the binomial generalizes the Poisson.

# What might such a random variable be?

- Probability generating function $P_n(y) = 1 - D_n(y)^{1/n}$
  where $D_n(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$
- Presumably has something to do with random graphs on $n$ vertices
- Maybe some structure built on top of a random graph
  (some kind of tree? Markov chain?)

Try to understand the first two cases:

$$\begin{aligned} P_2(y) &= 1 - (1-y)^{1/2} \\ &= \tfrac{1}{2}y + \tfrac{1}{8}y^2 + \tfrac{1}{16}y^3 + \tfrac{5}{128}y^4 + \tfrac{7}{256}y^5 + \tfrac{21}{1024}y^6 \\ &\qquad + \tfrac{33}{2048}y^7 + \tfrac{429}{32768}y^8 + \tfrac{715}{65536}y^9 + \tfrac{2431}{262144}y^{10} + \ldots \\ &\sim \text{Sibuya}(\tfrac{1}{2}) \text{ random variable} \end{aligned}$$

$$\begin{aligned} P_3(y) &= 1 - (1 - \tfrac{3}{2}y + \tfrac{1}{2}y^3)^{1/3} \\ &= \tfrac{1}{2}y + \tfrac{1}{4}y^2 + \tfrac{1}{24}y^3 + \tfrac{1}{24}y^4 + \tfrac{1}{48}y^5 + \tfrac{5}{288}y^6 \\ &\qquad + \tfrac{7}{576}y^7 + \tfrac{23}{2304}y^8 + \tfrac{329}{41472}y^9 + \tfrac{553}{82944}y^{10} + \ldots \end{aligned}$$

How are these related to random graphs on 2 or 3 vertices?

I have an analytic proof that $P_3(y) \succeq 0$, but it doesn't shed any light on the possible probabilistic interpretation.

Jim Fill has a probabilistic interpretation for $n = 2, 3$ in terms of birth-and-death chains, but it doesn't seem to generalize to $n \geq 4$.

## Ratios of roots $x_k(y)/x_{k+1}(y)$

The series

$$\frac{x_0(y)}{x_1(y)} \;=\; \tfrac{1}{2}y \,+\, \tfrac{1}{6}y^2 \,+\, \tfrac{5}{72}y^3 \,+\, \tfrac{11}{216}y^4 \,+\, \tfrac{29}{1296}y^5 \,+\, \ldots$$

has nonnegative coefficients at least up to order $y^{136}$.
(But its reciprocal does not have any fixed signs.)

**Big Conjecture #7.** The series $x_0(y)/x_1(y)$ has all nonnegative coefficients.

**Consequence of Big Conjecture #7.** Since $\lim\limits_{y\uparrow 1} x_0(y)/x_1(y) = 1$, Big Conjecture #7 implies that $|x_0(y)| < |x_1(y)|$ for all $y \in \mathbb{D}$ (a special case of Big Conjecture #2 on the separation in modulus of roots).

- But unfortunately … the series

  $$\frac{x_1(y)}{x_2(y)} \;=\; \tfrac{2}{3}y \,+\, \tfrac{1}{18}y^2 \,+\, \tfrac{17}{216}y^3 \,+\, \tfrac{23}{810}y^4 \,+\, \tfrac{343}{17280}y^5 \,+\, \ldots$$

  has a negative coefficient at order $y^{13}$. This doesn't contradict the conjecture that $|x_1(y)/x_2(y)| < 1$ in the unit disc, but it does rule out the simplest method of proof.

**NEED TO DO:** Use Schur algorithm to test $|x_1(y)/x_2(y)| < 1$ to higher order. Then extend to $x_k(y)/x_{k+1}(y)$.

## Asymptotics of roots as $y \to 1$

Write $y = e^{-\gamma}$ with $\operatorname{Re}\gamma > 0$.
Want to study $\gamma \to 0$ (non-tangentially in the right half-plane).

I *believe* I will be able to prove that

$$-x_k(e^{-\gamma}) \;\approx\; \frac{1}{e}\gamma^{-1} \,+\, c_k\gamma^{-1/3} \,+\, \ldots$$

for suitable constants $c_0 < c_1 < c_2 < \ldots$ . But I have not yet worked out all the details.

**Overview of method:**

1. Develop an asymptotic expansion for $F(x, e^{-\gamma})$ when $\gamma \to 0$ and $x$ is taken to be of order $\gamma^{-1}$, because this is the regime where the zeros will be found.
2. Use this expansion for $F(x, e^{-\gamma})$ to deduce an expansion for $x_k(e^{-\gamma})$.

**Sketch of step #1:** Insert Gaussian integral representation for $e^{-\frac{\gamma}{2}n^2}$ to obtain

$$F(x, e^{-\gamma}) \;=\; (2\pi\gamma)^{-1/2} \int_{-\infty}^{\infty} \exp[g(t)]\, dt$$

with

$$g(t) \;=\; -\frac{t^2}{2\gamma} \,+\, xe^{\gamma/2}e^{it}$$

Saddle-point equation $g'(t) = 0$ is $-ite^{-it} = \gamma e^{\gamma/2}x$, so it makes sense to make the change of variables

$$x = \gamma^{-1}e^{-\gamma/2}we^{w} ,$$

which puts the saddle point at $t_0 = iw$. (Note that this brings in the Lambert $W$ function, i.e. the inverse function to $w \mapsto we^w$.) We then have

$$F(\gamma^{-1}e^{-\gamma/2}we^{w}, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \int_{-\infty}^{\infty} dt \, \exp\left[-\frac{t^2}{2\gamma} + \frac{we^w}{\gamma}e^{it}\right]$$

Now shift the contour to go through the saddle point (parallel to the real axis) and make the change of variables $t = s + iw$: we have

$$F(\gamma^{-1}e^{-\gamma/2}we^{w}, e^{-\gamma}) = (2\pi\gamma)^{-1/2} \exp\left[\frac{w^2}{2\gamma} + \frac{w}{\gamma}\right] \int_{-\infty}^{\infty} ds \, \exp[h(s)]$$

where

$$h(s) = -\frac{(1+w)}{2\gamma}s^2 + \frac{w}{\gamma}\left(e^{is} - 1 - is + \frac{s^2}{2}\right)$$

and the integration goes along the real $s$ axis.

These formulae should allow computation of asymptotics

(a) $\gamma \to 0$ (in a suitable way) for (suitable values of) fixed $w$; and

(b) $w \to \infty$ (in a suitable direction) for (suitable values of) fixed $\gamma$.

Focus for now on (a).

Recall that

$$h(s) \;=\; -\frac{(1+w)}{2\gamma}s^2 \;+\; \frac{w}{\gamma}\Bigl(e^{is}-1-is+\frac{s^2}{2}\Bigr)$$

Consider for simplicity $\gamma$ and $x$ real. There seem to be three regimes:

- **"High temperature":** $w > -1$ (i.e. $we^w > -1/e$).

  Easiest case: $s = 0$ saddle point is Gaussian, and can compute the asymptotics to all orders in terms of 3-associated Stirling subset numbers $\left\{ n \atop m \right\}_{\ge 3}$. [Still need to justify this formal calculation by showing that only the $s = 0$ saddle point contributes.]

- **"Low temperature":** $w = -\eta\cot\eta + \eta i$ with $-\pi < \eta < \pi$ (i.e. $we^w < -1/e$).

  Saddle points at $s = 0$ and $s = 2\eta$ contribute; I *think* this is all.

- **"Critical regime":** $w = -(1+\xi\gamma^{1/3})$ with $\xi$ fixed, which corresponds to

  $$x \;=\; -\frac{1}{e\gamma}\left[1 \;-\; \frac{\xi^2}{2}\gamma^{2/3} \;+\; O(\gamma)\right]$$

  – *At* the "critical point" $\xi = 0$: Dominant behavior at $s = 0$ saddle point is no longer Gaussian (it vanishes) but rather the cubic term $is^3/(6\gamma)$. Can compute the asymptotics to all orders in terms of 4-associated Stirling subset numbers $\left\{ n \atop m \right\}_{\ge 4}$ (at least formally).

  – In the critical regime ($\xi$ arbitrary): Expect to have Airy asymptotics as in Flajolet–Salvy–Schaeffer (2004). This is where the roots will lie.

I would appreciate help with the details!!!

The polynomials $P_N(x,w) = \sum_{n=0}^{N} \binom{N}{n} x^n w^{n(N-n)}$

- Partition function of Ising model on complete graph $K_N$, with $x = e^{2h}$ and $w = e^{-2J}$
- Related to binomial $(1+x)^N$ in same way as our $F(x,y)$ is related to exponential $e^x$
  [but we have written $w^{n(N-n)}$ instead of $y^{n(n-1)/2}$]
- $\lim_{N\to\infty} P_N\!\left(\frac{xw^{1-N}}{N},\, w\right) \;=\; F(x, w^{-2}) \quad \text{when } |w| > 1$
- So results about zeros of $P_N$ generalize those about $F$ (just as results about the binomial generalize those about the exponential function)
- Lee–Yang theorem: In ferromagnetic case ($0 \le w \le 1$), all zeros are on the unit circle $|x| = 1$
- Laguerre: In antiferromagnetic case ($w \ge 1$), all zeros are real and negative
- What about "complex antiferromagnetic" case $|w| > 1$??

**Big Conjecture #8.** For $|w| > 1$, all zeros of $P_N(\,\cdot\,, w)$ are separated in modulus (by at least a factor $|w|^2$).

Taking $N \to \infty$, this implies Big Conjecture #2 about the separation in modulus of the zeros of $F(\,\cdot\,, y)$.

# Differential-equation approach to $P_N(x,w) = \sum_{n=0}^{N} \binom{N}{n} x^n w^{n(N-n)}$

On the space of polynomials $Q_N(x) = \sum_{n=0}^{N} a_n x^n$ of degree $N$ with $a_0 \neq 0$, define the semigroup

$$(\mathcal{A}_t Q_N)(x) \;\equiv\; \sum_{n=0}^{N} a_n x^n e^{tn(N-n)}$$

Roots of $\mathcal{A}_t Q_N$ evolve according to an *autonomous* differential equation, which is best expressed in terms of *logarithms* of roots $\zeta_i = \log x_i$:

$$\frac{d\zeta_i}{dt} \;=\; \sum_{j \neq i} f(\zeta_i - \zeta_j)$$

where

$$f(z) \;=\; \coth(z/2)$$

These are first-order ("Aristotelian") equations of motion for a system of $n$ "particles" (in $\mathbb{R}$ or $\mathbb{C}$) with a translation-invariant "force" $f$.

Moreover, the specific force $f = \coth$ is a Calogero–Moser–Sutherland system, much studied in the theory of integrable systems.

For polynomials $Q_N$ with *real* roots and *real* $t > 0$, this approach gives interesting results on separation of zeros. (In particular, it gives a new proof of Laguerre's theorem.)

Is this approach useful for *complex* $t$ with $\mathrm{Re}\, t > 0$???
Can it be used to prove Big Conjecture #8?

## A more general approach to the leading root $x_0(y)$ (When stumped, generalize ...!)

Consider a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$$

normalized to $\alpha_0 = \alpha_1 = 1$, or more generally

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y) \, x^n$$

where

(a) $a_0(0) = a_1(0) = 1$;

(b) $a_n(0) = 0$ for $n \ge 2$; and

(c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$.

**Examples:**

- The "partial theta function"

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2}$$

- The "deformed exponential function"

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$$

- More generally, consider

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n \, y^{n(n-1)/2}}{(1+q)(1+q+q^2) \, \cdots \, (1+q+\ldots+q^{n-1})}$$

  which reduces to $\Theta_0$ when $q = 0$, and to $F$ when $q = 1$.

- "Deformed binomial" and "deformed hypergeometric" series (see below).

# A more general approach, continued ...

- A power series for the leading root $x_0(y)$ can be computed from the power-series expansion of $\log f(x,y)$. This is extremely efficient! (see next lecture)
- Example: For $\Theta_0$ we have

  $$-x_0(y) \;=\; 1+y+2y^2+4y^3+9y^4+21y^5+52y^6+133y^7+351y^8+\ldots$$

  with strictly positive coefficients at least through order $y^{6999}$.
- More generally, for $\widetilde{R}(x,y,q)$ it can be proven that

  $$-x_0(y,q) \;=\; 1 \,+\, \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)}\, y^n$$

  where

  $$Q_n(q) \;=\; \prod_{k=2}^{\infty} (1+q+\ldots+q^{k-1})^{\lfloor n/\binom{k}{2} \rfloor}$$

  and $P_n(q)$ is a self-inversive polynomial with integer coefficients.

  I have verified for $n \le 349$ that $P_n(q)$ has *two* interesting positivity properties:

  (a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]\, P_5(q) = 0$.

  (b) $P_n(q) > 0$ for $q > -1$.

Can any of this be proven???

YES!!! … A first teeny breakthrough …

… but please stay tuned for our next installment …

## A general approach to the leading root $x_0(y)$

- Start from a formal power series

$$f(x, y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n$$

  where

  (a) $a_0(0) = a_1(0) = 1$
  (b) $a_n(0) = 0$ for $n \geq 2$
  (c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$

  and coefficients lie in a commutative ring-with-identity-element $R$.

- By (c), each power of $y$ is multiplied by only *finitely many* powers of $x$.

- That is, $f$ is a formal power series in $y$ whose coefficients are *polynomials* in $x$, i.e. $f \in R[x][[y]]$.

- Hence, for *any* formal power series $X(y)$ with coefficients in $R$ [not necessarily with zero constant term], the composition $f(X(y), y)$ makes sense as a formal power series in $y$.

- Not hard to see (by the implicit function theorem for formal power series or by a direct inductive argument) that there exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$.

- We call $x_0(y)$ the **leading root** of $f$.

- Since $x_0(y)$ has constant term $-1$, we will write $x_0(y) = -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.

How to compute $\xi_0(y)$?

1. **Elementary method:** Insert $\xi_0(y) = 1 + \sum_{n=1}^{\infty} b_n y^n$ into $f(-\xi_0(y), y) = 0$ and solve term-by-term.

2. Method based on the explicit implicit function formula (see below).

3. Method based on the exponential formula and expansion of $\log f(x, y)$ (see again below).

- Method #3 is computationally very efficient. (It's what I used above.)

- Method #2 gives an *explicit* formula for the coefficients of $\xi_0(y)$ ...
- Can it also be used to give *proofs*?

# Tools I: The explicit implicit function formula

- See A.D.S., arXiv:0902.0069 or Stanley, vol. 2, Exercise 5.59
- (Almost trivial) generalization of Lagrange inversion formula
- Comes in analytic-function and formal-power-series versions
- Recall Lagrange inversion: If $f(x) = \sum_{n=1}^{\infty} a_n x^n$ with $a_1 \neq 0$ (as either analytic function or formal power series), then

$$f^{-1}(y) \;=\; \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] \biggl( \frac{\zeta}{f(\zeta)} \biggr)^{m}$$

where $[\zeta^n] g(\zeta)$ denotes the coefficient of $\zeta^n$ in the power series $g(\zeta)$. More generally, if $h(x) = \sum_{n=0}^{\infty} b_n x^n$, we have

$$h(f^{-1}(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] \, h'(\zeta) \biggl( \frac{\zeta}{f(\zeta)} \biggr)^{m}$$

- Rewrite this in terms of $g(x) = x/f(x)$: then $f(x) = y$ becomes $x = g(x)y$, and its solution $x = \varphi(y) = f^{-1}(y)$ is given by the power series

$$\varphi(y) \;=\; \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] g(\zeta)^m$$

and

$$h(\varphi(y)) \;=\; h(0) \,+\, \sum_{m=1}^{\infty} \frac{y^m}{m} \, [\zeta^{m-1}] h'(\zeta) g(\zeta)^m$$

## The explicit implicit function formula, continued

- Generalize $x = g(x)y$ to $x = G(x, y)$, where
  - $G(0, 0) = 0$ and $|(\partial G/\partial x)(0, 0)| < 1$ (analytic-function version)
  - $G(0, 0) = 0$ and $(\partial G/\partial x)(0, 0) = 0$ (formal-power-series version)
- Then there is a unique $\varphi(y)$ with zero constant term satisfying $\varphi(y) = G(\varphi(y), y)$, and it is given by

  $$\varphi(y) \;=\; \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}] G(\zeta, y)^m$$

  More generally, for any $H(x, y)$ we have

  $$H(\varphi(y), y) \;=\; H(0, y) \,+\, \sum_{m=1}^{\infty} \frac{1}{m}\,[\zeta^{m-1}] \frac{\partial H(\zeta, y)}{\partial \zeta}\, G(\zeta, y)^m$$

- Proof imitates standard proof of the Lagrange inversion formula: the variables $y$ simply “go for the ride”.
- Alternate interpretation: Solving fixed-point problem for the family of maps $x \mapsto G(x, y)$ parametrized by $y$. Variables $y$ again “go for the ride”.

## Application to leading root of $f(x, y)$

- Start from a formal power series $f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$ satisfying properties (a)–(c) above.
- Write out $f(-\xi_0(y), y) = 0$ and add $\xi_0(y)$ to both sides:

$$\xi_0(y) \;=\; a_0(y) \;-\; [a_1(y) - 1]\xi_0(y) \;+\; \sum_{n=2}^{\infty} a_n(y)\, (-\xi_0(y))^n$$

- Insert $\xi_0(y) = 1 + \varphi(y)$ where $\varphi(y)$ has zero constant term. Then $\varphi(y) = G(\varphi(y), y)$ where

$$G(z, y) \;=\; \sum_{n=0}^{\infty} (-1)^n\, \widehat{a}_n(y)\, (1+z)^n$$

and

$$\widehat{a}_n(y) \;=\; \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1 \\ a_n(y) & \text{for } n \ge 2 \end{cases}$$

And $\varphi(y)$ is the *unique* formal power series with zero constant term satisfying this fixed-point equation.

- Since this $G$ satisfies $G(0,0) = 0$ and $(\partial G/\partial z)(0,0) = 0$ [indeed it satisfies the stronger condition $G(z, 0) = 0$], we can apply the explicit implicit function formula to obtain an explicit formula for $\xi_0(y)$:

$$\xi_0(y) \;=\; 1 \;+\; \sum_{m=1}^{\infty} \frac{1}{m}\, [\zeta^{m-1}] \left( \sum_{n=0}^{\infty} (-1)^n\, \widehat{a}_n(y)\, (1+\zeta)^n \right)^{m}$$

More generally, for any formal power series $H(z, y)$, we have

$$\begin{aligned} &H(\xi_0(y) - 1, y) \\ &= \; H(0, y) \;+\; \sum_{m=1}^{\infty} \frac{1}{m}\, [\zeta^{m-1}] \frac{\partial H(\zeta, y)}{\partial \zeta} \left( \sum_{n=0}^{\infty} (-1)^n\, \widehat{a}_n(y)\, (1+\zeta)^n \right)^{m} \end{aligned}$$

## Application to leading root of $f(x, y)$, continued

- In particular, by taking $H(z, y) = (1+z)^\beta$ we can obtain an explicit formula for an arbitrary power of $\xi_0(y)$:

$$\xi_0(y)^\beta \;=\; 1 + \sum_{m=1}^{\infty} \frac{\beta}{m} \sum_{n_1,\ldots,n_m \ge 0} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} (-1)^{n_i} \widehat{a}_{n_i}(y)$$

- Important special case: $a_0(y) = a_1(y) = 1$ and $a_n(y) = \alpha_n \, y^{\lambda_n}$ $(n \ge 2)$ where $\lambda_n \ge 1$ and $\lim\limits_{n\to\infty} \lambda_n = \infty$. Then

$$[y^N] \frac{\xi_0(y)^\beta - 1}{\beta} \;=\; \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{n_1, \ldots, n_m \ge 2 \\ \sum\limits_{i=1}^{m} \lambda_{n_i} = N}} (-1)^{\sum n_i} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} \alpha_{n_i}$$

- Can this formula be used for proofs of nonnegativity???
- *Empirically* I know that the RHS is $\ge 0$ when $\lambda_n = n(n-1)/2$:
  - For $\beta \ge -2$ with $\alpha_n = 1$ (partial theta function)
  - For $\beta \ge -1$ with $\alpha_n = 1/n!$ (deformed exponential function)
  - For $\beta \ge -1$ with $\alpha_n = (1-q)^n/(q;q)_n$ and $q > -1$
- And I can *prove* this (by a *different* method!) for the partial theta function (but not yet for the others).
- **How can we see these facts from this formula???**
  [open combinatorial problem]

## Tools II: Variants of the exponential formula

- Let $R$ be a commutative ring containing the rationals.
- Let $A(x) = \sum_{n=0}^{\infty} a_n\, x^n$ be a formal power series (with coefficients in $R$) satisfying $a_0 = 1$.
- Now consider $C(x) = \log A(x) = \sum_{n=1}^{\infty} c_n\, x^n$.
- It is well known (and easy to prove) that

$$a_n \;=\; \sum_{k=1}^{n} \frac{k}{n}\, c_k\, a_{n-k} \qquad \text{for } n \ge 1$$

  This allows $\{a_n\}$ to be calculated given $\{c_n\}$, or vice versa.
- Sometimes useful to introduce $\widetilde{C}_n = nc_n$, which are the coefficients in the logarithmic derivative

$$\frac{x\, A'(x)}{A(x)} \;=\; \sum_{n=1}^{\infty} \widetilde{C}_n\, x^n$$

- See Scott–Sokal, arXiv:0803.1477 for generalizations to $A(x)^\lambda$ and applications to the multivariate Tutte polynomial

- Now specialize to $R = R_0[[y]]$ and $A(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$ where $a_0(y) = 1$
- Assume further that $a_1(0) = 1$ and $a_n(0) = 0$ for $n \ge 2$ [conditions (a) and (b) for our $f(x,y)$]
- Then

$$\frac{x\, A'(x,y)}{A(x,y)} \;=\; \sum_{n=1}^{\infty} \widetilde{C}_n(y)\, x^n$$

  where $'$ denotes $\partial/\partial x$ and $\widetilde{C}_n(y)$ has constant term $(-1)^{n-1}$.

## Application to leading root of $f(x,y)$

- Start from a formal power series $f(x,y) = 1 + x + \sum\limits_{n=2}^{\infty} a_n(y)\, x^n$ satisfying
$$a_n(y) \;=\; O(y^{\alpha(n-1)}) \qquad \text{for } n \geq 2$$
for some real $\alpha > 0$. [This is a bit stronger than (a)–(c).]

- Define $\{\widetilde{C}_n(y)\}_{n=1}^{\infty}$ by
$$\frac{x\, f'(x,y)}{f(x,y)} \;=\; \sum_{n=1}^{\infty} \widetilde{C}_n(y)\, x^n$$
where $'$ denotes $\partial/\partial x$.

- **Theorem:** We have
$$\widetilde{C}_n(y) \;=\; (-1)^{n-1}\, \xi_0(y)^{-n} \;+\; O(y^{\alpha n})$$
or equivalently
$$\xi_0(y) \;=\; [(-1)^{n-1} \widetilde{C}_n(y)]^{-1/n} \;+\; O(y^{\alpha n})$$

- This theorem provides an extraordinarily efficient method for computing the series $\xi_0(y)$:
  - Compute the $\widetilde{C}_n(y)$ inductively using the recursion
  $$\widetilde{C}_n \;=\; n a_n \;-\; \sum_{k=1}^{n-1} \widetilde{C}_k\, a_{n-k}$$
  - Take the power $-1/n$ to extract $\xi_0(y)$ through order $y^{\lceil \alpha n \rceil - 1}$

- This abstracts the recursive method first used for the special case
$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}\,.$$

## Proof of Theorem (via complex analysis)

- Use complex-analysis argument to prove Theorem when $R = \mathbb{C}$ and $f$ is a polynomial.
- Infer general validity by some abstract nonsense.

**Lemma.** Fix a real number $\alpha > 0$, and let $P(x, y) = 1 + x + \sum_{n=2}^{N} a_n(y) x^n$ where the $\{a_n(y)\}_{n=2}^{N}$ are polynomials with complex coefficients satisfying $a_n(y) = O(y^{\alpha(n-1)})$. Then there exist numbers $\rho > 0$ and $\gamma > 0$ such that $P(\,\cdot\,, y)$ has precisely one root in the disc $|x| < \gamma |y|^{-\alpha}$ whenever $|y| \le \rho$.

**Idea of proof of Lemma:** Apply Rouché's theorem to $f(x) = x$ and $g(x) = 1 + \sum_{n=2}^{N} a_n(y) x^n$ on the circle $|x| = \gamma |y|^{-\alpha}$.

**Proof of Theorem when $R = \mathbb{C}$ and $f$ is a polynomial:** Write

$$P(x, y) \;=\; \prod_{i=1}^{k(y)} \Bigl(1 - \frac{x}{X_i(y)}\Bigr)$$

with $k(y) \le N$. Therefore

$$\frac{x\, P'(x, y)}{P(x, y)} \;=\; \sum_{i=1}^{k(y)} \frac{-x/X_i(y)}{1 \,-\, x/X_i(y)}$$

and hence

$$\widetilde{C}_n(y) \;=\; -\sum_{i=1}^{k(y)} X_i(y)^{-n} \;.$$

Now, for small enough $|y|$, one of the roots is given by the *convergent* series $-\xi_0(y)$ and is smaller than $\gamma |y|^{-\alpha}$ in magnitude, while the other roots have magnitude $\geq \gamma |y|^{-\alpha}$ by the Lemma. We therefore have

$$\left|\widetilde{C}_n(y) \;-\; (-1)^{n-1}\xi_0(y)^{-n}\right| \;\leq\; (N-1)\gamma^{-n}|y|^{\alpha n}$$

for small enough $|y|$, as claimed. $\square$

**Proof of Theorem in general case:** Write

$$a_n(y) \;=\; \sum_{m=\lceil \alpha(n-1)\rceil}^{\infty} a_{nm}\, y^m$$

Work in the ring $R = \mathbb{Z}[\boldsymbol{a}]$ where $\boldsymbol{a} = \{a_{nm}\}_{n\geq 2,\, m\geq \lceil \alpha(n-1)\rceil}$ are treated as indeterminates. Then the claim of the Theorem amounts to a series of identities between polynomials in $\boldsymbol{a}$ with integer coefficients. We have verified these identities when evaluated on collections $\boldsymbol{a}$ of complex numbers of which only finitely many are nonzero; and this is enough to prove them as identities in $\mathbb{Z}[\boldsymbol{a}]$. $\square$

There is also a direct formal-power-series proof (due to Ira Gessel) at least in the case $\alpha = 1$. I don't know whether it extends to arbitrary real $\alpha > 0$.

More recently, Jim Fill and I have discovered an even simpler formal-power-series proof, based on Waring's (1762) formula for the power-sum symmetric functions in terms of the elementary symmetric functions.

**General philosophy.** Some positivity properties (proven or conjectured) for $\xi_0(y)$ may actually hold for $[(-1)^{n-1}\widetilde{C}_n(y)]^{-1/n}$ for each finite $n$.

## Computational use of Theorem

- Can compute $\xi_0(y)$ through order $y^{N-1}$ by computing $\widetilde{C}_N(y)$
- Do this by computing $\widetilde{C}_n(y)$ for $1 \le n \le N$ using recursion
- Observe that all $\widetilde{C}_n(y)$ can be truncated to order $y^{N-1}$
  [no need to keep the full polynomial of degree $n(n-1)/2$]

- For $F$, have done $N = 900$
  [$N = 400$ takes a minute, $N = 900$ takes less than 6 hours;
  but $N = 900$ needs 24 GB memory!]

- For $\Theta_0$, have done $N = 7000$
  [$N = 500$ takes a minute, $N = 1500$ takes less than an hour;
  $N = 7000$ took 11 days and 21 GB memory]

- For $\widetilde{R}$, have done $N = 350$
  [$N = 50$ takes a minute, $N = 100$ takes less than an hour;
  $N = 350$ took a month and 10 GB memory]

## Some positivity properties of formal power series

- Consider formal power series with real coefficients

$$f(y) \;=\; 1 + \sum_{m=1}^{\infty} a_m \, y^m$$

- For $\alpha \in \mathbb{R}$, define the class $\mathcal{S}_\alpha$ to consist of those $f$ for which

$$\frac{f(y)^\alpha - 1}{\alpha} \;=\; \sum_{m=1}^{\infty} b_m(\alpha) \, y^m$$

  has all nonnegative coefficients (with a suitable limit when $\alpha = 0$).

- In other words:
  - For $\alpha > 0$ (resp. $\alpha = 0$), the class $\mathcal{S}_\alpha$ consists of those $f$ for which $f^\alpha$ (resp. $\log f$) has all nonnegative coefficients.
  - For $\alpha < 0$, the class $\mathcal{S}_\alpha$ consists of those $f$ for which $f^\alpha$ has all *nonpositive* coefficients after the constant term 1.

- Containment relations among the classes $\mathcal{S}_\alpha$ are given by the following fairly easy result:

  **Proposition** (Scott–A.D.S., unpublished):
  Let $\alpha, \beta \in \mathbb{R}$. Then $\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ if and only if either

  (a) $\alpha \le 0$ and $\beta \ge \alpha$, or
  (b) $\alpha > 0$ and $\beta \in \{\alpha, 2\alpha, 3\alpha, \ldots\}$.

  Moreover, the containment is strict whenever $\alpha \neq \beta$.

## Application to deformed exponential function $F$

As shown earlier, it seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}
\xi_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\
& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
& + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\
& + \ldots + \text{terms through order } y^{899}
\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}
\xi_0(y)^{-1} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
& - \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\
& - \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\
& - \ldots - \text{terms through order } y^{899}
\end{aligned}$$

**But I have no proof of either of these conjectures!!!**

- Note that $\xi_0(y)$ is analytic on $0 \le y < 1$ and diverges as $y \uparrow 1$ like $1/[e(1-y)]$.
- It follows that $\xi_0(y) \notin \mathcal{S}_\alpha$ for $\alpha < -1$.

## Application to partial theta function $\Theta_0$

It seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}\xi_0(y) &= 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ &\quad +948y^9 + 2610y^{10} + \ldots + \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}\xi_0(y)^{-1} &= 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ &\quad -178y^9 - 490y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-2}$:

$$\begin{aligned}\xi_0(y)^{-2} &= 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ &\quad -138y^9 - 386y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

Here I *do* have a proof of these properties (see below).

- Note that
$$\frac{\xi_0(y)^\alpha - 1}{\alpha} = y + \frac{\alpha+3}{2}y^2 + \frac{(\alpha+2)(\alpha+7)}{6}y^3 + O(y^4)$$
- So $\xi_0(y) \notin \mathcal{S}_\alpha$ for $\alpha < -2$.

Application to $\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(1+q)\,\cdots\,(1+q+\ldots+q^{n-1})}$

- Can use explicit implicit function formula to prove that

$$\xi_0(y;q) \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)}\, y^n$$

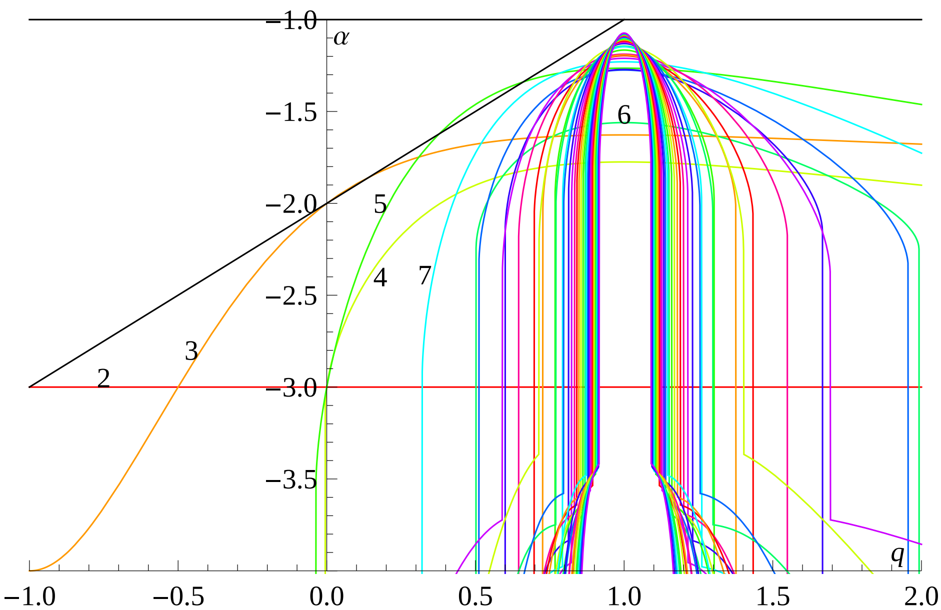
where

$$Q_n(q) \;=\; \prod_{k=2}^{\infty} (1+q+\ldots+q^{k-1})^{\lfloor n/\binom{k}{2}\rfloor}$$

and $P_n(q)$ is a self-inversive polynomial in $q$ with integer coefficients.

- *Empirically* $P_n(q)$ has *two* interesting positivity properties:
  (a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]\, P_5(q) = 0$.
  (b) $P_n(q) > 0$ for $q > -1$.

- *Empirically* $\xi_0(y;q) \in \mathcal{S}_{-1}$ for all $q > -1$:




- **Can any of this be proven for $q \neq 0$?**

## The deformed binomial series

Here is an even simpler family that interpolates between the partial theta function $\Theta_0$ and the deformed exponential function $F$:

- Start from the Taylor series for the binomial $f(x) = (1-\mu x)^{-1/\mu}$
  [it is convenient to parametrize it in this way]
  and introduce factors $y^{n(n-1)/2}$ as usual:

$$
\begin{aligned}
F_\mu(x,y) &= \sum_{n=0}^{\infty} (-\mu)^n \binom{-1/\mu}{n} x^n \, y^{n(n-1)/2} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \prod_{j=0}^{n-1} (1+j\mu) \right) x^n \, y^{n(n-1)/2}
\end{aligned}
$$

- We call $F_\mu(x,y)$ the *deformed binomial function*.
- For $\mu = 0$ it reduces to the deformed exponential function.
- For $\mu = 1$ it reduces to the partial theta function.
- For $\mu = -1/N$ $(N = 1, 2, 3, \ldots)$ it is a polynomial of degree $N$ that is the "$y$-deformation" of the binomial $(1 + x/N)^N$

## The deformed binomial series, continued

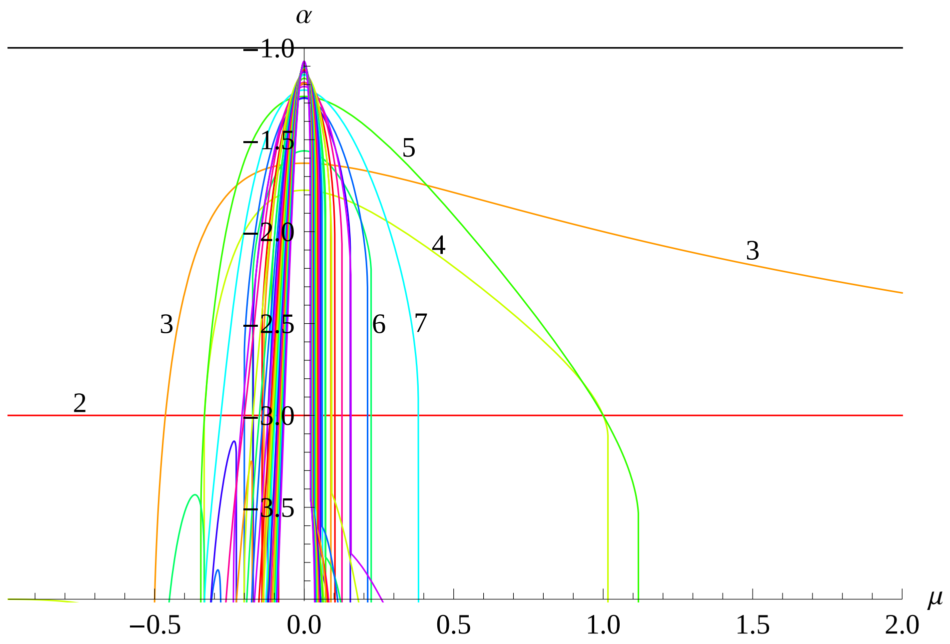
- Can use explicit implicit function formula to prove that

$$\xi_0(y;\mu) \;=\; 1 + \sum_{n=1}^{\infty} \frac{P_n(\mu)}{d_n}\, y^n$$

where $P_n(\mu)$ is a polynomial of degree $n$ with integer coefficients and $d_n$ are explicit integers.

- *Empirically* $P_n(\mu)$ has *two* interesting positivity properties:
  (a) $P_n(\mu)$ has all strictly positive coefficients.
  (b) $P_n(\mu) > 0$ for $\mu > -1$.

- *Empirically* $\xi_0(y;\mu) \in \mathcal{S}_{-1}$ for all $\mu > -1$:




- **Can any of this be proven for $\mu \neq 1$?**

## The deformed hypergeometric series

- Exponential (${}_0F_0$) and binomial (${}_1F_0$) are simplest cases of the hypergeometric series ${}_pF_q$.
- Can apply "$y$-deformation" process to ${}_pF_q$:

$$ {}_pF_q^* \left( \begin{matrix} \mu_1, \ldots, \mu_p \\ \nu_1, \ldots, \nu_q \end{matrix} \middle| \, x, y \right) \;=\; \sum_{n=0}^{\infty} \frac{(1;\mu_1)^{\overline{n}} \, \cdots \, (1;\mu_p)^{\overline{n}}}{(1;\nu_1)^{\overline{n}} \, \cdots \, (1;\nu_q)^{\overline{n}}} \, \frac{x^n}{n!} \, y^{n(n-1)/2} $$

where

$$ (1;\mu)^{\overline{n}} \;=\; \prod_{j=0}^{n-1} (1 + j\mu) $$

- Note that setting $\mu_p = 0$ reduces ${}_pF_q^*$ to ${}_{p-1}F_q^*$ (and likewise for $\nu_q$).
- *Empirically* the two positivity properties for the deformed binomial appear to extend to ${}_2F_0^*$ (in the two variables $\mu_1, \mu_2$).
- I expect that this will generalize to all ${}_pF_0^*$.

- But the cases ${}_pF_q^*$ with $q \geq 1$ are different, and I do not yet know the complete pattern of behavior.

## Identities for the partial theta function

- Use standard notation for $q$-shifted factorials:

$$
\begin{aligned}
(a;q)_n &= \prod_{j=0}^{n-1}(1-aq^j) \\
(a;q)_\infty &= \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{for } |q|<1
\end{aligned}
$$

- A pair of identities for the partial theta function:

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (y;y)_\infty\, (-x;y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n\, (-x;y)_n} \\
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (-x;y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\, (-x;y)_n}
\end{aligned}
$$

  as formal power series and as analytic functions on $(x,y) \in \mathbb{C} \times \mathbb{D}$

- Rewrite these as

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (y;y)_\infty\, (-xy;y)_\infty \left[1+x+\sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\, (-xy;y)_{n-1}}\right] \\
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (-xy;y)_\infty \left[1+x+\sum_{n=1}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\, (-xy;y)_{n-1}}\right]
\end{aligned}
$$

- The first identity goes back to Heine (1847).

- The second identity can be found in Andrews and Warnaar (2007) but is probably much older.

## Proof that $\xi_0 \in \mathcal{S}_1$ for the partial theta function

- Let's say we use the first identity:

$$\Theta_0(x,y) \;=\; (y;y)_\infty\,(-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\,(-xy;y)_{n-1}}\right]$$

- So $\Theta_0(x,y) = 0$ is equivalent to "brackets = 0".
- Insert $x = -\xi_0(y)$ and bring $\xi_0(y)$ to the LHS:

$$\xi_0(y) \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n}(1-y^j)\,\prod\limits_{j=1}^{n-1}[1-y^j\xi_0(y)]}$$

- This formula can be used iteratively to determine $\xi_0(y)$, and in particular to prove the strict positivity of its coefficients:
- Define the map $\mathcal{F}\colon\ \mathbb{Z}[[y]] \to \mathbb{Z}[[y]]$ by

$$(\mathcal{F}\xi)(y) \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n}(1-y^j)\,\prod\limits_{j=1}^{n-1}[1-y^j\xi(y)]}$$

- Define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{Z}[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$.
- Then $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \ldots \preceq \xi_0$ and $\xi_0^{(k)}(y) \;=\; \xi_0(y) \;+\; O(y^{3k+1})$.
- In particular, $\lim\limits_{k\to\infty} \xi_0^{(k)}(y) = \xi_0(y)$, and $\xi_0(y)$ has strictly positive coefficients.
- Thomas Prellberg (arXiv:1210.0095) has a combinatorial interpretation of $\xi_0(y)$ and $\xi_0^{(k)}(y)$.
- Proofs of $\xi_0 \in \mathcal{S}_{-1}$ and $\xi_0 \in \mathcal{S}_{-2}$ use second identity in a similar way.

## A conjectured big picture

I conjecture that there are *three different* things going on here:

- **Positivity properties for the leading root $\boldsymbol{\xi_0(y)}$:**
  - $\xi_0(y)$ in various classes $\mathcal{S}_\beta$ for a fairly large class of series $f(x,y) = \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$
  - Appears to include deformed hypergeometric ${}_pF_0^*$, Rogers–Ramanujan $\widetilde{R}(x,y,q)$, probably others
  - Find sufficient conditions on $\{\alpha_n\}_{n=0}^{\infty}$??

- **Positivity properties for the higher roots $\boldsymbol{\xi_k(y)}$:**
  - *Some* positivity for partial theta function and perhaps others (needs further investigation)
  - Positivity of *all* $\xi_k(y)$ *only* for deformed exponential??

- **Positivity properties for ratios $\boldsymbol{\xi_k(y)/\xi_{k+1}(y)}$:**
  - Holds for some unknown class of series $f(x,y)$
  - Even for polynomials, class is unknown (cf. Calogero–Moser): roots should be "not too unevenly spaced"
  - Class appears to include at least deformed exponential
  - Needs much further investigation

# Summary of open questions

- All the Big Conjectures concerning $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$.

- For a formal power series
$$f(x,y) \;=\; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$$
with $\alpha_0 = \alpha_1 = 1$, find simple *sufficient* conditions to have $\xi_0(y) \succeq 0$ or more generally $\xi_0(y) \in \mathcal{S}_\beta$.

- In particular, want to handle $\alpha_n = 1/n!$ or $\alpha_n = (1-q)^n/(q;q)_n$ or $\alpha_n = (-\mu)^n \binom{-1/\mu}{n}$ or hypergeometric generalizations.

- Can this be done using explicit implicit function formula?
  (open combinatorial problem)

- Understand positivity properties for higher roots $x_k(y)$ and ratios of roots $x_k(y)/x_{k+1}(y)$.

- Understand the first-order Calogero–Moser–Sutherland system
$$\frac{d\zeta_i}{dt} \;=\; \sum_{j \neq i} f(\zeta_i - \zeta_j)$$
with $f(z) = 1/z$ [roots of polynomial solution of 1D heat equation] or $f(z) = \coth z$, especially at *complex* time $t$.