Polynomial Chaos and Scaling Limits of Disordered Systems

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Joint work
with

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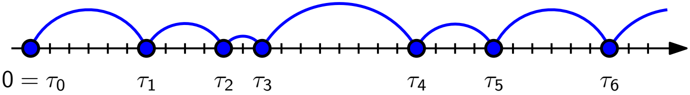
Nikos Zygouras (Warwick)

Outline

1. Disordered Systems (Disorder Relevance vs Irrelevance)
   - Disordered Pinning Model
   - Long-range Directed Polymer Model
   - Random Field Ising Model
2. Disorder Relevance via Continuum and Weak Disorder Limits
   - Polynomial chaos expansions for partition functions
   - Lindeberg Principle for polynomial chaos expansions
   - Convergence of polynomial chaos to Wiener chaos expansions
3. Some Open Questions
4. Marginal Disorder Relevance

1.1 The Homogeneous Pinning Model



Let $\tau := \{\tau_0 = 0 < \tau_1 < \tau_2 \cdots\} \subset \mathbb{N}_0$ be a recurrent renewal process, with law $\mathbf{P}$, and

$$\mathbf{P}(\tau_1 = n) \sim \frac{C}{n^{1+\alpha}} \qquad \text{for some exponent } \alpha > 0.$$

The Pinning Model is defined by the family of Gibbs measures:

$$\mathbf{P}_{N,h}(\tau) = \frac{1}{Z_{N,h}} e^{h \sum_{n=1}^{N} 1_{\{n \in \tau\}}} \mathbf{P}(\tau) \qquad (\text{expectation } \mathbf{E}_{N,h}[\cdot]),$$

where N is the system size, $h \in \mathbb{R}$ determines the interaction strength, and $Z_{N,h} = \mathbf{E}[e^{h \sum_{n=1}^{N} 1_{\{n \in \tau\}}}]$ is the partition function.

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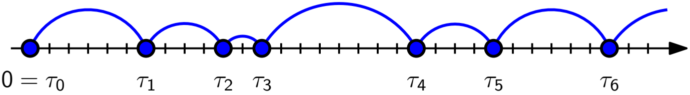

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1.2 Phase Transition for the Pinning Model

As h varies, the pinning model undergoes a localization-delocalization transition. More precisely, there is a critical h_c $(= 0)$ such that

- For $h < h_c$, the limiting contact fraction

$$g(h) := \lim_{N\to\infty} \mathbf{E}_{N,h}\Big[\frac{1}{N}\sum_{n=1}^{N} 1_{\{n\in\tau\}}\Big] = 0;$$

- For $h > h_c$, the limiting contact fraction $g(h) > 0$.

Furthermore, $g(h) = F'(h)$, where the free energy

$$F(h) = \lim_{N\to\infty} \frac{1}{N}\log Z_{N,h} \begin{cases} = \quad 0 & \text{if } h \le h_c, \\ \approx C(h-h_c)^{\gamma} & \text{as } h \downarrow h_c. \end{cases}$$

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We now add disorder.

Let $\omega := (\omega_n)_{n\in\mathbb{N}}$ be i.i.d. with $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[e^{\lambda\omega_1}] < \infty$ for all λ close to 0.

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There exists $\hat{h}_c(\beta) < 0$, s.t. for $\mathbb{P}$-a.e. ω, the contact fraction

$$\hat{g}(\beta, h) := \lim_{N\to\infty} \mathbb{E}\mathbf{E}^{\omega}_{N,\beta,h}\Big[\frac{1}{N}\sum_{n=1}^{N} 1_{\{n\in\tau\}}\Big] \begin{cases} = 0 & \text{if } h < \hat{h}_c(\beta), \\ > 0 & \text{if } h > \hat{h}_c(\beta). \end{cases}$$

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Basic Question: Does disorder modify the qualitative nature of the homogeneous model (without disorder)?

For the pinning model, we say that disorder is

- relevant if the critical exponents $\hat{\gamma}(\beta) \neq \gamma$ for all $\beta > 0$ (no matter how weak is the disorder strength);
- irrelevant if $\hat{\gamma}(\beta) = \gamma$ for $\beta > 0$ sufficiently small.

For the pinning model with renewal exponent α, it has been shown:

- Disorder is relevant for $\alpha > \frac{1}{2}$;
- Disorder is irrelevant for $\alpha < \frac{1}{2}$;
- Disorder is marginally relevant for $\alpha = \frac{1}{2}$.

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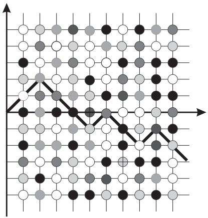
2.1 Directed Polymer Model



Let $X := (X_n)_{n\in\mathbb{N}_0}$ be a mean-zero random walk on $\mathbb{Z}^d$ with law $\mathbf{P}$.

Let $\omega := (\omega(n,x))_{n\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ be i.i.d. with $\mathbb{E}[\omega(0,o)] = 0$, and $\mathbb{E}[e^{\lambda\omega(0,o)}] < \infty$ for all λ close to 0.

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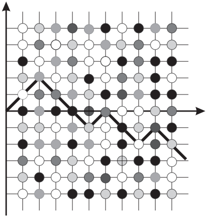

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There exists a critical $\beta_c = \beta_c(d) \geq 0$, such that if X is a diffusive random walk on $\mathbb{Z}^d$, then

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Assuming X to be diffusive, it has been shown that:

- $\beta_c(d) = 0$ for $d = 1$ and 2, and hence disorder is relevant;
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Assuming that $d = 1$ and X is in the domain of attraction of an α-stable process for some $\alpha \in (0, 2]$, then similarly:

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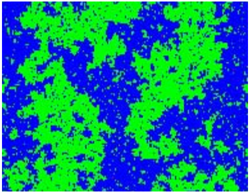

The Ising model on a domain $\Omega \subset \mathbb{Z}^d$ with $+$ boundary condition, at inverse temperature $\beta \geq 0$ and external field $h \in \mathbb{R}$, is given by the following Gibbs measure on spin configurations $(\sigma_x)_{x\in\Omega} \in \{\pm 1\}^\Omega$:

$$\mathbf{P}^+_{\Omega,\beta,h}(\sigma) = \frac{1}{Z^+_{\Omega,\beta,h}} \exp\left\{\beta \sum_{x\sim y\in\Omega\cup\partial\Omega} \sigma_x\sigma_y + h\sum_{x\in\Omega}\sigma_x\right\}\mathbf{P}(\sigma)$$

where $\mathbf{P}$ is the uniform distribution on $\{\pm 1\}^\Omega$, and $Z^+_{\Omega,\beta,h}$ is the partition function. The free energy is defined by

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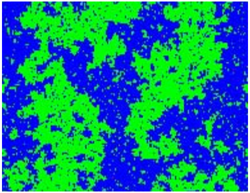

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For $d = 2$, $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$, and as we vary the external field h at $\beta = \beta_c$, Camia-Garban-Newman'12 recently showed that

$$m(\beta_c, h) = \Theta(h^{\frac{1}{15}}) \qquad \text{as } h \downarrow 0.$$

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3.3 The Two-Dimensional Random Field Ising Model

We now add disorder to the Ising model on $\mathbb{Z}^2$ at $\beta = \beta_c$ in the form of a random external field.

Let $\omega := (\omega_x)_{x\in\mathbb{Z}^2}$ be i.i.d. with $\mathbb{E}[\omega_x] = 0$ and $\mathbb{E}[e^{\lambda\omega_x}] < \infty$ for all λ close to 0.

Given ω, disorder strength $\lambda \geq 0$ and external field $h \in \mathbb{R}$, we define the Random Field version of the critical Ising model on $\Omega \subset \mathbb{Z}^2$ by

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where $Z^{\omega}_{\Omega,\lambda,h}$ is the partition function.

Question: Is disorder relevant in the sense that for arbitrary small disorder strength $\lambda > 0$, the magnetization

$$\hat{m}(\lambda,h) := \lim_{\Omega\uparrow\mathbb{Z}^2} \mathbb{E}\mathbf{E}^{\omega}_{\Omega,\lambda,h}\Big[\frac{1}{|\Omega|}\sum_{x\in\Omega}\sigma_x\Big] \approx Ch^{\gamma} \quad \text{as } h \downarrow 0$$

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Question: Is disorder relevant in the sense that for arbitrary small disorder strength $\lambda > 0$, the magnetization

$$\hat{m}(\lambda, h) := \lim_{\Omega\uparrow\mathbb{Z}^2} \mathbb{E}\mathbf{E}^{\omega}_{\Omega,\lambda,h}\Big[\frac{1}{|\Omega|}\sum_{x\in\Omega}\sigma_x\Big] \approx Ch^{\gamma} \quad \text{as } h \downarrow 0$$

for some critical exponent $\gamma \neq \frac{1}{15}$? (Belief: $\gamma > \frac{1}{15}$.)

3.3 The Two-Dimensional Random Field Ising Model

We now add disorder to the Ising model on $\mathbb{Z}^2$ at $\beta = \beta_c$ in the form of a random external field.

Let $\omega := (\omega_x)_{x\in\mathbb{Z}^2}$ be i.i.d. with $\mathbb{E}[\omega_x] = 0$ and $\mathbb{E}[e^{\lambda\omega_x}] < \infty$ for all λ close to 0.

Given ω, disorder strength $\lambda \geq 0$ and external field $h \in \mathbb{R}$, we define the Random Field version of the critical Ising model on $\Omega \subset \mathbb{Z}^2$ by

$$\mathbf{P}^{\omega}_{\Omega,\lambda,h}(\sigma) = \frac{1}{Z^{\omega}_{\Omega,\lambda,h}} \exp\Big\{\sum_{x\in\Omega}(\lambda\omega_x + h)\sigma_x\Big\}\mathbf{P}^{+}_{\Omega,\beta_c,0}(\sigma),$$

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4.1 Disorder Relevance via Scaling Limits (Heuristics)

We propose a new perspective on disorder relevance/irrelevance, which gives a unified treatment for many disordered systems.

Observation: Disorder relevance means: fixed disorder strength, however weak, is still too strong since it changes the qualitative features of the homogeneous model in the ∞-volume limit.

To moderate the effect of disorder, it should be possible to tune the strength of disorder down to zero as the system size tends to infinity (while rescaling space), so that disorder persists in such a continuum and weak disorder limit.

Thus disorder relevance manifests itself in the existence of a non trivial continuum disordered model in a suitable weak disorder and continuum limit. (Consistent with Harris' Criterion'74).

Inspired by Alberts-Khanin-Quastel'12 construction of the Continuum Directed Polymer Model in dimension $1 + 1$, we cast things in the general framework of disorder relevance-irrelevance, give general criteria for convergence to continuum disordered models, and apply them to new models of interest.

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4.2 The General Setting

We first study continuum and weak disorder limits of the partition function in a general setting, which incorporates all previous models.

Let $\Omega \subset \mathbb{R}^d$. For $\delta \in (0,1)$, let $\Omega_\delta := \Omega \cap (\delta\mathbb{Z})^d$. Let $(\omega_x)_{x\in\Omega_\delta}$ be i.i.d. with $\mathbb{E}[\omega_x] = 0$ and $\mathbb{E}[e^{\lambda\omega_x}] < \infty$ for all λ close to 0.

Let $\mathbf{P}_{\Omega_\delta}$ be a probability measure on $(\sigma_x)_{x\in\Omega_\delta} \in \{0,1\}^{\Omega_\delta}$ that defines the homogeneous reference model. Given ω, disorder strength λ and bias h, add disorder in the form of a random field by defining

$$\mathbf{P}^\omega_{\Omega_\delta,\lambda,h}(\sigma) = \frac{1}{Z^\omega_{\Omega_\delta,\lambda,h}} e^{\sum_{x\in\Omega_\delta}(\lambda\omega_x+h)\sigma_x}\mathbf{P}_{\Omega_\delta}(\sigma),$$

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To identify non-trivial disordered limits of $Z^\omega_{\Omega_\delta,\lambda,h}$ in the continuum and weak disorder limit $\delta \downarrow 0$, $\lambda = \lambda(\delta) \downarrow 0, h = h(\delta) \downarrow 0$, we first rewrite $Z^\omega_{\Omega_\delta,\lambda,h}$ in a polynomial chaos expansion.

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4.3 Polynomial Chaos Expansion for Partition Function

Because $\sigma_x \in \{0,1\}$, by Mayer cluster expansion,

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\begin{aligned}
Z^{\omega}_{\Omega_\delta,\lambda,h} &= \mathbf{E}_{\Omega_\delta}\Big[\prod_{x\in\Omega_\delta} e^{(\lambda\omega_x+h)\sigma_x}\Big] \\
&= \mathbf{E}_{\Omega_\delta}\Big[\prod_{x\in\Omega_\delta} (1+\xi_x\sigma_x)\Big] \qquad (\xi_x := e^{\lambda\omega_x+h}-1) \\
&= 1+\sum_{k=1}^{\infty}\sum_{\substack{I=\{x_1,\ldots,x_k\}\subset\Omega_\delta \\ |I|=k}} \mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}]\,\xi_{x_1}\cdots\xi_{x_k},
\end{aligned}
$$

which is multi-linear in the i.i.d. random variables $(\xi_x)_{x\in\Omega_\delta}$ with

$$
\mathbb{E}[\xi_x] \approx h(\delta) + \frac{\lambda^2(\delta)}{2} =: \tilde{h}(\delta), \qquad \mathbb{V}ar(\xi_x) \approx \lambda^2(\delta) \quad \text{as } \delta \downarrow 0.
$$

Each ξ_x is associated with a cube Δ_x of side length δ in $(\delta\mathbb{Z})^d$, and we can replace ξ_x by a normal variable with the same mean and variance

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\xi_x \longrightarrow \int_{\Delta_x} \lambda(\delta)\delta^{-\frac{d}{2}} W(\mathrm{d}u) + \int_{\Delta_x} \tilde{h}(\delta)\delta^{-d}\mathrm{d}u,
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where $W(\mathrm{d}u)$ is a d-dimensional white noise on $\mathbb{R}^d$. This is justified by a Lindeberg principle, extending Mossel-O'Donnell-Oleszkiewicz'10.

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4.4 Convergence to Wiener Chaos Expansions

We then have

$$Z^{\omega}_{\Omega_\delta,\lambda,h} \overset{\delta\downarrow 0}{\approx} 1+\sum_{k=1}^{\infty}\frac{1}{k!}\int\cdots\int_{\Omega^k} \mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}]\prod_{i=1}^{k}\left(\lambda\delta^{-\frac{d}{2}}W(\mathrm{d}x_i)+\tilde{h}\delta^{-d}\mathrm{d}x_i\right).$$

Key Assumption: $\exists\gamma\geq 0$ s.t. the rescaled k-point correlation function

$$(\delta^{-\gamma})^k\mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}] \xrightarrow[\delta\downarrow 0]{L^2} \psi_\Omega(x_1,\ldots,x_k)\in L^2(\Omega^k),$$

which entails that

$$\psi_\Omega(x_1,\ldots,x_k)\approx\|x_i-x_j\|^{-\gamma}\quad\text{as } x_i\to x_j,$$

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Remark. For $\mathbf{P}_{\Omega_\delta}$ defined as a Gibbs measure, the Key Assumption implies that the reference model $\mathbf{P}_{\Omega_\delta}$ is at the critical point of a continuous phase transition.

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Then by the assumed convergence of the k-point correlation functions,

$$Z^{\omega}_{\Omega_\delta,\lambda,h} \overset{\delta\downarrow 0}{\approx}$$

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \delta^{-k\gamma} \mathbf{E}_{\Omega_\delta}[\sigma_{x_1} \cdots \sigma_{x_k}] \prod_{i=1}^{k} \big(\lambda\delta^{\gamma-\frac{d}{2}} W(\mathrm{d}x_i) + \tilde{h}\delta^{\gamma-d}\mathrm{d}x_i\big)$$

$$\underset{\delta\to 0}{\Longrightarrow} \mathcal{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}} := 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi_\Omega(x_1, \ldots, x_k) \prod_{i=1}^{k} \big(\hat{\lambda} W(\mathrm{d}x_i) + \hat{h}\mathrm{d}x_i\big),$$

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Remark. The above approach fails when $\gamma = d/2$, which is called the marginal case and includes the pinning model with $\alpha = 1/2$, the short-range directed polymer in $\mathbb{Z}^{2+1}$, and the long-range directed polymer in $\mathbb{Z}^{1+1}$ with $\alpha = 1$.

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$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \delta^{-k\gamma} \mathbf{E}_{\Omega_\delta}[\sigma_{x_1} \cdots \sigma_{x_k}] \prod_{i=1}^{k} \big(\lambda\delta^{\gamma-\frac{d}{2}} W(\mathrm{d}x_i) + \tilde{h}\delta^{\gamma-d}\mathrm{d}x_i\big)$$

$$\underset{\delta\to 0}{\Longrightarrow} \mathcal{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}} := 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int_{\Omega^k} \psi_\Omega(x_1, \ldots, x_k) \prod_{i=1}^{k} \big(\hat{\lambda} W(\mathrm{d}x_i) + \hat{h}\mathrm{d}x_i\big),$$

which is a Wiener-chaos expansion w.r.t. a white noise with possibly non-zero mean (the expansion may diverge in L^2!).

Remark. The above approach fails when $\gamma = d/2$, which is called the marginal case and includes the pinning model with $\alpha = 1/2$, the short-range directed polymer in $\mathbb{Z}^{2+1}$, and the long-range directed polymer in $\mathbb{Z}^{1+1}$ with $\alpha = 1$.

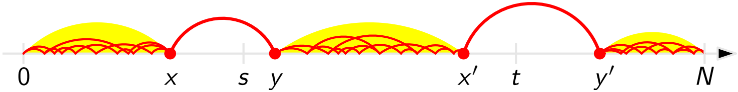
$\Omega := [0,1]$, and $\mathbf{P}_{\Omega_\delta}$ is the law of the rescaled renewal process. Then

$$(\delta^{\min\{1,\alpha\}-1})^k \mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}] \xrightarrow[\delta\downarrow 0]{L^2} \psi(x_1,\ldots,x_k),$$

where ψ is the correlation function of the α-stable regenerative set and is in L^2 exactly when $\alpha > \frac{1}{2}$ (disorder relevant regime). Let

$$\lambda(\delta) = \hat{\lambda}\delta^{\min\{1,\alpha\}-\frac{1}{2}}, \qquad h(\delta) = \hat{h}\delta^{\min\{1,\alpha\}} - \lambda^2(\delta)/2.$$

Then the partition function $Z^\omega_{\Omega_\delta,\lambda,h}$ converges weakly to $\mathcal{Z}^W_{\Omega,\hat{\lambda},\hat{h}}$.



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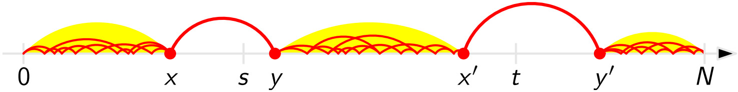

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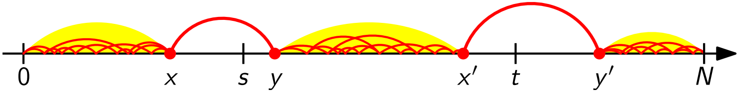

4.6 Scaling Limit of the Long-range Directed Polymer

Let $\Omega := [0,1] \times \mathbb{R}$, and let $\Omega_\delta := \Omega \cap (\delta\mathbb{Z}) \times (\delta^{1/\alpha}\mathbb{Z})$ with $\alpha \in (0,2]$. Let $\mathbf{P}_{\Omega_\delta}$ be the law of a rescaled random walk, which converges in distribution to an α-stable process as $\delta \downarrow 0$. Then

$$(\delta^{-1/\alpha})^k \mathbf{E}_{\Omega_\delta}[\sigma_{(t_1,x_1)} \cdots \sigma_{(t_k,x_k)}] \xrightarrow[\delta\downarrow 0]{L^2} \psi((t_1,x_1),\ldots,(t_k,x_k)),$$

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Then the random partition function $Z^\omega_{\Omega_\delta,\lambda}$ converges weakly to $\mathcal{Z}^W_{\Omega,\hat{\lambda}}$, generalizing work of Alberts-Khanin-Quastel'12 for the case $\alpha = 2$.

Extending the weak convergence to the family of point-to-point partition functions $(Z^{\omega,c}_\lambda(s,x;t,y))_{0\le s<t\le 1; x,y\in\mathbb{R}}$, we obtain a family of continuum partition functions $(\mathcal{Z}^{W,c}_{\hat{\lambda}}(s,x;t,y))_{0\le s<t\le 1; x,y\in\mathbb{R}}$, which can be used to construct the Continuum Long-range Directed Polymer, extending Alberts-Khanin-Quastel'12.

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Constructing a Continuum Random Field Ising Model out of $\mathcal{Z}^W_{\Omega,\hat{\lambda},\hat{h}}$ seems highly non-trivial, although very interesting. It is expected to be a generalized field, as in the case with no disorder ($\lambda = 0$) constructed recently by Camia-Garban-Newman'13.

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5.2 Universality for Long-range Directed Polymer

For each $\alpha \in (1, 2]$, by taking the continuum and weak disorder limit, we can construct a family of disordered point-to-point continuum partition functions $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, x)$.

As a function in $t \geq 0$ and $x \in \mathbb{R}$, $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, x)$ is a mild solution for the stochastic fractional heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta^{\frac{\alpha}{2}} u + \hat{\lambda} W u, \\ u(0, \cdot) = \delta_0(\cdot). \end{cases}$$

For $\alpha = 2$, as $\hat{\lambda} : 0 \uparrow \infty$, the distribution of $\log \mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, 0)$ is known to smoothly interpolate between the Gaussian and the Tracy-Widom GUE distribution, which gives the universal fluctuation of short-range directed polymers in $\mathbb{Z}^{1+1}$.

Question: For $\alpha \in (1, 2)$, as $\hat{\lambda} \uparrow \infty$, does the law of $\log \mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, 0)$ converge to a limit that generalizes Tracy-Widom GUE and governs the universal fluctuation of α-stable directed polymer in $\mathbb{Z}^{1+1}$?

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5.3 Open Questions for Random Field Ising

- Go beyond the partition function and construct the Continuum Random Field Ising Model as a generalized random field in a white noise environment (extending Camia-Garban-Newman'13 for the non-disordered case). The law of the disordered field is likely singular w.r.t. the non-disordered field. Such an object is similar in spirit to solutions of singular SPDEs constructed via Hairer's theory of regularity structures.

- Since the partition functions of the random field perturbation of the critical Ising model on $\mathbb{Z}^2$ has non-trivial disordered limits, it is natural to conjecture that disorder is relevant in the sense that:

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