Outline

1 CoLoR
   - Background: termination of rewriting
   - Motivation
   - CoLoR architecture
   - History
   - Overview
   - Related work
   - Certified competition

2 Formalization of matrix interpretations
   - Introduction to matrix interpretations
   - Monotone algebras
   - Matrices
   - Matrix interpretations
   - Practicalities
Introduction to term rewriting

Example (Plus)

Let’s define plus in Peano arithmetic.

\[
\begin{align*}
0 + y & \rightarrow y \\
\text{s}(x) + y & \rightarrow \text{s}(x + y)
\end{align*}
\]

Example (Computing with plus)

Now let us do some maths... how about \(2 + 2\)?

\[
\begin{align*}
\text{s}(\text{s}(0)) + \text{s}(\text{s}(0)) & \rightarrow \text{s}(\text{s}(0) + \text{s}(\text{s}(0))) \\
\text{s}(\text{s}(0 + \text{s}(\text{s}(0)))) & \rightarrow \text{s}(\text{s}(\text{s}(\text{s}(0))))
\end{align*}
\]

Definition

A TRS is **terminating** iff it does not admit infinite reductions.
Termination of rewriting:

- is undecidable.
- is an important topic in term rewriting.
- Many methods exist and new ones are constantly being developed.
- Recently the emphasis is on automation.
- There exists a number of tools for proving termination.
- Stimulated by an annual termination competition.
- Tools (and proofs that they produce) are getting more and more complex.
Motivation

- Certification of results of termination provers.
- Common proof format for termination provers:
  - common tools (proof presentation, manipulation, dots),
  - control language for provers (integration of tools)
- Extension of proof assistance kernels.
CoLoR approach to termination

How to certify termination results?

- Possibility: certification of tools source code.
  ⇒ difficult, tool dependent, extra work with every change, . . .
- CoLoR approach:
  - TPG: common format for termination proofs.
  - Tools output proofs in TPG format.
  - CoLoR: a Coq library of results on termination.
  - Rainbow: a tool for translation from proofs in TPG format to Coq proofs, using results from CoLoR.
CoLoR architecture overview

Termination provers

APoVE  TTT  TORPA

Rainbow

Certified termination techniques

RPO  Polynomial interpretations  Semantic labelling

CoLoR

Coq

proof.v

Library

proof.xml
History

- Project started (Blanqui)  March 2004
- First release  March 2005
- First certified proofs  July 2006
- First certification workshop  May 2007
- First certified competition  June 2007
Termination criteria:
- **matrix interpretations** [Koprowski, Zantema]
- dependency graph cycles [Blanqui]
- higher-order recursive path ordering [Koprowski]
- recursive path ordering [Coupet-Grimal, Delobel]
- multiset ordering [Koprowski]
- **polynomial interpretations** [Hinderer]

Transformation techniques:
- dependency pairs [Blanqui]
- rule elimination [Blanqui]
- arguments filtering [Blanqui]
- conversion from algebraic to varyadic terms [Blanqui]
Content of CoLoR.

- General libraries:
  - matrices [Koprowski]
  - simply typed lambda-terms [Koprowski]
  - finite multisets [Koprowski]
  - varyadic terms [Blanqui]
  - algebraic terms with symbols of fixed arity [Hinderer, Blanqui]
  - integer polynomials with multiple variables [Hinderer]
  - vectors [Hinderer, Blanqui]
  - lists, relations, etc.
Size of CoLoR

- 42,000 lines of code.
- Half of the size of Coq standard library.
- 5% of Coq contribs.

Structure:
- Terms: 44%
- Data structures: 29%
- Termination criteria: 17%
- Mathematical structures: 10%

Coq constructs:
- Inductive definitions: 38
- Recursive functions: 116
- Non-recursive definitions: 560
- Lemmas and theorems: 2170
Related work

- **CoLoR project**
  - Authors: Blanqui, . . .
  - Tool: TPA, . . .
  - Proof assistant: Coq

- **A3PAT project**
  - Authors: Contejean, . . .
  - Tool: CiME
  - Proof assistant: Coq

- **Isabelle/HOL termination checker**
  - Authors: Bulwahn, Krauss, Nipkow, . . .
  - Tool: TTT
  - Proof assistant: Isabelle/HOL
Certified competition

- In the termination competition this year a new “certified” category introduced.

- Participants:
  - CiME + A3PAT
  - TPA + CoLoR
  - TTT₂ + CoLoR

- Many questions remain, like
  - Who’s the winner?
  - Competition VS Cooperation
Termination competition

![Graph showing the number of problems solved over years for different systems: AProVE, Jambox, TPA, CiME, Matchbox, TEPARLA, MU-TERM, TTTbox, TTT, CoLoR (comp.), and CoLoR (now). The years range from 2004 to 2007. The number of problems solved ranges from 0 to 400.}
Example

z086.trs

\[ a(a(x)) \rightarrow c(b(x)), \quad b(b(x)) \rightarrow c(a(x)), \quad c(c(x)) \rightarrow b(a(x)) \]

Matrix interpretation for z086.trs

\[
\begin{align*}
a(x) &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
b(x) &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
c(x) &= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]
Example ctd.

Termination proof for z086.trs

\[ a(a(x)) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \]

\[ c(b(x)) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left( \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \]
Monotone algebras

Definition (Monotonicity)
An operation \([f] : A \times \cdots \times A \to A\) is \textit{monotone} with respect to a binary relation \(\succ\) on \(A\) if
\[
a_i \succ a'_i \implies [f](a_1, \ldots, a_i, \ldots a_n) \succ [f](a_1, \ldots, a'_i, \ldots, a_n).
\]

Definition
Given a relation \(\succ\) on \(A\) we define its extension to a relation on terms as:
\[
s \succ_T t \equiv \forall \alpha : \mathcal{X} \to A, [s, \alpha] \succ [t, \alpha]
\]
Definition (A weakly monotone $\Sigma$-algebra)

A *weakly monotone $\Sigma$-algebra* $(A, [\cdot], >, \gtrsim)$ is a $\Sigma$-algebra $(A, [\cdot])$ equipped with two binary relations $>$, $\gtrsim$ on $A$ such that

- $>$ is well-founded;
- $> \cdot \gtrsim \subseteq >$;
- for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $\gtrsim$.

Definition (An *extended monotone* $\Sigma$-algebra)

An *extended monotone $\Sigma$-algebra* $(A, [\cdot], >, \gtrsim)$ is a weakly monotone $\Sigma$-algebra $(A, [\cdot], >, \gtrsim)$ in which moreover for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $>$.
Monotone algebras

Theorem

Let $R, R', S, S'$ be TRSs over a signature $\Sigma$, $(\mathcal{A}, [\cdot], >, \succsim)$ be an extended monotone $\Sigma$-algebra such that:

- $\ell \succsim_T r$ for every rule $\ell \rightarrow r$ in $R \cup S$ and
- $\ell >_T r$ for every rule $\ell \rightarrow r$ in $R' \cup S'$

Then $\text{SN}(R/S)$ implies $\text{SN}(R \cup R' / S \cup S')$.

Theorem

Let $R, R', S, S'$ be TRSs over a signature $\Sigma$, let $(\mathcal{A}, [\cdot], >, \succsim)$ be a weakly monotone $\Sigma$-algebra such that:

- $\ell \succsim_T r$ for every rule $\ell \rightarrow r$ in $R \cup S$ and
- $\ell >_T r$ for every rule $\ell \rightarrow r$ in $R'$,

Then $\text{SN}(R_{\text{top}}/S)$ implies $\text{SN}((R \cup R')_{\text{top}}/S)$.
Monotone algebras are formalized as a functor.

Apart for the aforementioned requirements there is one additional required to deal with concrete examples: $>_T$ and $\preceq_T$ must be decidable.

More precisely the requirement is to provide a relation $\gg$, such that

- $\gg \subseteq >_T$ and
- $\gg$ is decidable
- similarly for $\preceq$.

The structure returned by the functor contains all the machinery required to prove (relative)-(top)-termination in Coq.
Matrices are formalized as a functor taking as an argument the semi-ring of coefficients $\mathcal{R}$ and providing a structure of matrices of arbitrary sizes with coefficients in $\mathcal{R}$ and a number of basic operations over matrices such as:

\[
[\cdot], \quad M_{i,j}, \quad M + N, \quad M \cdot N, \quad M^T, \ldots
\]

and a number of basic properties such as:

- $M + N = N + M$,
- $M \cdot (N \cdot P) = (M \cdot N) \cdot P$
- monotonicity of $\cdot$
- $\ldots$
Polynomial interpretations in the setting of monotone algebras

- $A = \mathbb{Z}$,
- $> = >_{\mathbb{Z}}$, $\geq = \geq_{\mathbb{Z}}$,
- interpretations represented by polynomials $[f(x_1, \ldots, x_n)] = P_{\mathbb{Z}}(x_1, \ldots, x_n)$,
- $>_{\mathcal{T}}$ not decidable (positiveness of polynomial) — heuristics required.
Matrix interpretations in the setting of monotone algebras

- fix a dimension $d$,
- $A = \mathbb{N}^d$,
- $(u_1, \ldots, u_d) \succeq (v_1, \ldots, v_d)$ iff $\forall i, u_i \geq_{\mathbb{N}} v_i$,
- $(u_1, \ldots, u_d) > (v_1, \ldots, v_d)$ iff $(u_1, \ldots, u_d) \succeq (v_1, \ldots, v_d) \land u_1 >_{\mathbb{N}} v_1$,
- interpretations represented as:
  $[f(x_1, \ldots, x_n)] = M_1 x_1 + \ldots + M_n x_n + v$
  where $M_i \in \mathbb{N}^{d \times d}$, $v \in \mathbb{N}^d$,
- $>_{\mathcal{T}}$ and $\succeq_{\mathcal{T}}$ are decidable in this case but thanks to introducing $\gg$ we do not need to prove completeness of their characterization.
- Domain fixed to $\mathbb{N}$ with natural orders $>$ and $\geq$. 
Formalization size (LOC):
- Monotone algebras: 351
- Matrices: 642
- Matrix interpretations: 673
- Polynomial interpretations in MA setting: 116
Evaluation of **TPA + Rainbow** on TPDB 3.2 (864 TRSs):

- polynomial interpretations: 167
- matrix interpretations: 237
- polynomial and matrix interpretations: 275

  - Verification time: AVG: 5sec. MAX: 75sec.
  - Proof steps: AVG: 5 MAX: 29

- polynomial and matrix interpretation in the DP setting: 379
Figure: Now
Thank you for your attention.