Recent developments in proving termination of rewriting automatically

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Prose, September 21, 2006
Term rewriting is a natural and basic framework to describe computations

Example:

Natural numbers are terms composed from the constant 0 and the unary symbol s
Term rewriting

Binary operations $+$ and $\ast$ are defined on these natural numbers by the following rules:

- $0 + x \rightarrow x$
- $s(x) + y \rightarrow s(x + y)$
- $0 \ast x \rightarrow 0$
- $s(x) \ast y \rightarrow y + (x \ast y)$
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\]

Applying these rules as long as possible (rewriting to normal form) always yields the desired result:

\[
\underbrace{s(s(0)) \ast s(s(s(0))))}_{x} \rightarrow \underbrace{s(s(s(0)))) + (s(0) \ast s(s(s(0))))}_{y} \rightarrow \cdots \rightarrow s(s(s(s(s(0)))))
\]
Why?
All rules are *sound*: the meaning of the left hand side is equal to the meaning of the right hand side

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All rules are *sound*: the meaning of the left hand side is equal to the meaning of the right hand side so by rewriting the meaning does not change.

For every ground term containing $\times$ or $+$ a rule is applicable

so if no rule is applicable then the term only consists of 0 and $s$

No *infinite* computations are possible

This latter is called *termination*, and that’s what we want to prove automatically.
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Most powerful transformational method: *dependency pairs* (Arts, Giesl, 2000)
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Since 2003, every year there is a competition comparing the power of these tools by applying them on a database of rewrite systems fully automatically. Characteristics of the 2006 competition:

- 11 participating tools
- Over 1000 rewrite systems
- Subdivided in 8 categories
- Time limit: 1 minute per tool per system (second round: 5 minutes)
- Total running time: two weeks
- Strongest tools: AProVE (Giesl et al, Aachen) and Jambox (Endrullis, VU Amsterdam), both written in JAVA.
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- The *matrix method* (Endrullis, Hofbauer, Waldmann, Zantema, 2006)
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**Definition (RPO)**

Given a precedence $\succ$ then $s \succ_{RPO} t$ iff

$s = f(s_1, \ldots, s_n)$ and

1. $s_i = t$ or $s_i \succ_{RPO} t$ for some $1 \leq i \leq n$, or
2. $t = g(t_1, \ldots, t_m)$, $s \succ_{RPO} t_i$ for all $1 \leq i \leq m$, and either
   - (a) $f \succ g$, or
   - (b) $f = g$ and $\langle s_1, \ldots, s_n \rangle \succ_{RPO} \langle t_1, \ldots, t_m \rangle$
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   (a) $f \succ g$, or
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**Theorem (RPO)**

*If $\succ$ is well-founded and $\ell \succ_{RPO} r$ for every rule $\ell \rightarrow r$ of a TRS $R$, then $R$ is terminating*
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- a way $\tau(f)$ to compare sequences of terms, for every symbol $f$
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Sometimes faster: introduce a boolean variable $p_{fg}$ for every pair of symbols $f, g$, representing whether $f \succ g$, and express the requirements in a SAT problem on these variables
Polynomials

Find an interpretation by strictly monotone polynomials such that by every rewrite rule the interpretation strictly decreases.
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*Example:*

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\begin{align*}
0 + x & \rightarrow x \\
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Interpretation [\cdot] in positive integers:

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[0] = 1, \ [s]x = x + 1, \ x[+]y = 2x + y, \ x[\ast]y = 3xy
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\[
[0][+]x = 2 + x > x \\
([s]x)[+]y = 2(x + 1) + y > 2x + y + 1 = [s](x[+]y) \\
[*]x = 3x > 1 = [0] \\
([s]x)[*]y = 3(x + 1) * y > 2y + 3xy = y[+] (x[*]y)
\]

for all $x, y > 0$, proving termination
Old approach for finding such interpretations: check requirements for all (or great number of randomly chosen) interpretations, typically
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- every constant is either 1, 2 or 8
- every unary symbol is either identity, successor, or multiply by 2
- every binary symbol is either addition, multiplication, or ...

Important: per symbol only a few options, otherwise the search space is intractable
New approach for finding such interpretations:
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- choose for constants: $A$
- choose for unary symbols $Ax + B$
- choose for binary symbols $Ax + By + C$
- ... 

where $A, B, C$ are unknown numbers represented in binary notation in $n$ bits
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If successful: transform the resulting satisfying assignment back to desired values $A, B, C, ...$
For this we need to be able to represent basic arithmetic: $+,\times,>,\cdots$ in propositional logic.
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This can be done straightforwardly, e.g. $a + b = d$:

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\begin{align*}
    c & \rightarrow 0 0 1 1 1 0 \\
    a = 7 & \rightarrow 0 0 1 1 1 1 \\
    b = 21 & \rightarrow 1 0 1 0 1 1 \\
    d = 28 & \rightarrow 1 1 1 0 0 0
\end{align*}
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\end{array}
\]

this is correct if and only if \(a_i \leftrightarrow b_i \leftrightarrow c_i \leftrightarrow d_i\) and
\(c_{i-1} \leftrightarrow ((a_i \land b_i) \lor (a_i \land c_i) \lor (b_i \land c_i))\)
for \(i = 1, \ldots, n\), and \(\neg c_0 \land \neg c_n\)
Multiplication can be expressed by repeated addition and duplication:
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\begin{align*}
  r &:= 0; \\
  \text{for } i &:= 1 \text{ to } n \text{ do} \\
  \quad &\text{begin} \\
  \quad &\quad s := 2 \times r; \\
  \quad &\quad \text{if } b_i \text{ then } r := s + a \text{ else } r := s \\
  \quad &\text{end}
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Executing this program yields \( r = a \times b \)
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All ingredients are easily expressed in propositional logic, so by introducing several fresh variables for representing intermediate values for \( r, s \) multiplication can be expressed in propositional logic.
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Final formula is conjunction of several small formulas
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So all ingredients of these polynomial interpretations transform to propositional logic

Final formula is conjunction of several small formulas

Writing these small formulas in CNF yields big CNF, on which SAT tools are directly applicable
The matrix method

Same idea as polynomials, except that now the interpretations are in \textit{vectors} over natural numbers rather than in natural numbers.
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Same idea as polynomials, except that now the interpretations are in *vectors* over natural numbers rather than in natural numbers.

Well-founded order:

$$(v_1, \ldots, v_d) > (u_1, \ldots, u_d) \iff v_1 > u_1 \land v_i \geq u_i \text{ for } i = 2, 3, \ldots, d$$
The matrix method

Same idea as polynomials, except that now the interpretations are in *vectors* over natural numbers rather than in natural numbers.

Well-founded order:

\[(v_1, \ldots, v_d) > (u_1, \ldots, u_d) \iff v_1 > u_1 \land v_i \geq u_i \text{ for } i = 2, 3, \ldots, d\]

Interpretation for symbol of arity \(k\):

\[ [f](\vec{x}_1, \ldots, \vec{x}_k) = A_1 \vec{x}_1 + \cdots + A_k \vec{x}_k + \vec{v} \]

where \(A_1, \ldots, A_k\) are \(d \times d\) matrices and \(\vec{v}\) is a vector, all over \(n\)-bits integers.
Typically $d \approx 3$ and $n \approx 3$
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Search space is huge, but everything can be expressed in SAT similarly: only arithmetic needed is $+, *, >$
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For small rewrite systems this easily yields formulas in 10,000 variables and over 100,000 clauses
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For SAT solvers like ZChaff, Minisat, SatElite this is no problem
Example:
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$$h(g(s(x), y), g(z, u)) \rightarrow h(g(u, s(z)), g(s(y), x))$$
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\[ h(g(s(x), y), g(z, u)) \rightarrow h(g(u, s(z)), g(s(y), x)) \]

\[
[h](\vec{x}_0, \vec{x}_1) = 
\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \cdot \vec{x}_0 + 
\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \cdot \vec{x}_1 + 
\begin{pmatrix} 0 \\ 2 \end{pmatrix}
\]

\[
[g](\vec{x}_0, \vec{x}_1) = 
\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \cdot \vec{x}_0 + 
\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \vec{x}_1
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[s](\vec{x}_0) = 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{x}_0 + 
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Conclusions

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SAT solving turns out to be helpful for proving termination of rewriting automatically, e.g. for
- recursive path order
- polynomial interpretations
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We gave a few examples of this, ignoring several crucial details like monotonicity requirements
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For the full power of these techniques combination with earlier techniques like dependency pairs and relative termination is essential
Are we done in this area?
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\[
\begin{align*}
f(t, x, y) & \rightarrow f(g(x, y), x, s(y)) \\
g(s(x), 0) & \rightarrow t \\
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Here \( x, y \) are variables, \( g \) stands for \textit{greater than} and \( t \) stands for \textit{true}

- second and third rule are the standard rules for \( > \) over the naturals composed from 0 and \( s \)(successor)
- The first rule describes the obviously terminating loop
  while \( x > y \) do \( y := y + 1 \)
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while \(x > y\) do \(y := y + 1\)

However, no tool can prove termination of this system.