Representation theory

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Theorem 1 (Frobenius, 1901)**.** *Let G be a group acting transitively on a finite set X such that for all $\sigma \in G \setminus \{1\}$ one has $\#\{x \in X : \sigma x = x\} \leq 1$. Then*

$$N = \{1\} \cup \{\sigma \in G : \forall x \in X : \sigma x \neq x\}$$

is a (normal) subgroup of G.

A group G is called a *Frobenius group* if an X and an action as in the theorem exist with $\#X \geq 2$ and the additional property that there are $\sigma \in G \setminus \{1\}$ and $x \in X$ with $\sigma x = x$; also, N is called the *Frobenius kernel* of G, and $\#X$ is called the *degree*.

Exercise L.1. Let G, X, N be as in the theorem of Frobenius, with $n = \#X \geq 2$.

(a) Prove: $\#N = n$.

(b) Suppose N is a subgroup. Prove: N is normal, and N acts transitively on X.

(c) Prove: $\#G = nd$ for some divisor d of $n - 1$.

Exercise L.2. Show by means of an example that the condition that X is finite cannot be omitted from Frobenius' theorem.

Exercise L.3. (a) Let R be a ring, $I \subset R$ a left ideal of finite index, and H a subgroup of the group R^* of units of R such that for all $a \in H \setminus \{1\}$ one has $R = (a-1)R + I$. Prove that $X = R/I$ and $G = \{\sigma : X \to X : \text{there exist } a \in H, b \in R : \text{for all } x \in R : \sigma(x \bmod I) = (ax + b \bmod I)\}$ satisfy the conditions of Frobenius' theorem. What is N?

(b) Show how to recover the examples D_n (n odd) from (a).

Exercise L.4. (a) Apply Exercise L.3 to the subring $R = \mathbb{Z}[i, j]$ of the division ring $\mathbb{H} = \mathbb{R} + \mathbb{R} \cdot i + \mathbb{R} \cdot j + \mathbb{R} \cdot ij$ of quaternions to construct a Frobenius group G of order $8 \cdot 9$ and degree 9 such that G contains the quaternion group $Q = \langle i, j \rangle$ of order 8.

(b) Apply Exercise L.3 to $R = \mathbb{Z}[i, (1 + i + j + ij)/2]$ to construct a Frobenius group of order $24 \cdot 25$ and degree 25 that contains Q.

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Exercise L.5*. Can you think of an example of a Frobenius group whose Frobenius kernel is non-abelian?