### Definition

Let $G$ be a group and $N$ a subgroup. Then $N$ is called *normal* if and only if $gN = Ng$ for all $g \in G$.

### Proposition

*If $N$ is a normal subgroup of $G$, then the product defined by $gN \cdot hN = ghN$ on $G/N$ defines a group on $G/N$.*

## Definition

A group $G$ is called *simple* if its only normal subgroups are

$$\{1\} \quad \text{and} \quad G.$$

## Example

An abelian simple group is cyclic of prime order.

## Example

The alternating group $A_n$, with $N \geq 5$ is simple.

If $G$ is a group, one can construct a tower of subgroups

$$\{1\} \lhd G_1 \lhd \cdots G_n \lhd G,$$

such that each quotient $G_{i+1}/G_i$ is simple.

The Jordan-Hölder Theorem states that the composition factors $G_{i+1}/G_i$ are (up to permutation of $i$ and isomorphism) uniquely determined.

## Definition

A group $G$ is called *solvable* if there is a sequence of subgroups

$$\{1\} \lhd G_1 \lhd \cdots G_n \lhd G,$$

such that each quotient $G_{i+1}/G_i$ is abelian.

## Example

The group $A_4$ is solvable.

## Example

The group of upper triangular invertible $n \times n$-matrices is solvable.

### Theorem

*Let $G$ be a finite group of prime power order, then $G$ is solvable.*

### Proof.

If $G$ is abelian, there is nothing to prove.
So, assume $G$ is not abelian.
$G$ acts on itself by conjugation. The size of a conjugacy class of an element $g \in G$ equals $|G|/|C_G(g)|$, which is 1 if $g \in Z(G)$ and divisible by $p$ otherwise.
This implies that the order of $Z(G)$ is divisible by $p$. In particular, $1 \neq Z(G) \neq G$.
Clearly, $Z(G)$ is solvable and, by induction, $G/Z(G)$ is solvable.
But then also $G$ is solvable. □

## Definition

Let $G$ be a finite group and $p$ a prime dividing the order of $G$.
A *Sylow $p$-subgroup* of $G$ is a subgroup of $G$ of order $p^n$, where $p^n$ is the highest power of $p$ dividing the order of $G$.

Figure: Ludwig Sylow, 1832-1918

## Theorem (Sylow, 1872)

*Let $G$ be a finite group and $p$ a prime dividing the order of $G$. Then the following hold:*

- *$G$ contains a Sylow $p$-subgroup;*
- *$G$ contains an element of order $p$;*
- *all Sylow $p$-subgroups of $G$ are conjugate;*
- *the number of Sylow $p$-subgroups divides $|G|$ and is $1$ modulo $p$.*

## Proposition

*Suppose $p$ and $q$ are prime. Let $G$ be a group of order $pq$. Then $G$ is solvable.*

## Proof.

If $p = q$, we can apply the previous result.
Suppose $p < q$. The number of Sylow $q$-subgroups is 1 mod $q$ and divides $p$. Thus, there is only one $q$-Sylow subgroup, $Q$ say.
The group $Q$ is normal in $G$. Both $Q$ and $G/Q$ are of prime power order, and hence solvable. Thus, also $G$ is solvable. □

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# Two famous results

Theorem (Burnside, 1897)

*Let $p$ and $q$ be two primes.*
*A group $G$ of order $p^\alpha q^\beta$ is solvable.*

Figure: William Burnside, 1852-1927

**Theorem (Feit-Thompson, The Odd Order Theorem, 1963)**

*A finite group of odd order is solvable.*

This result was conjectured by Burnside.

**Corollary**

*A non-abelian finite simple group contains an involution.*

The Odd Order Theorem is the starting point to the classification of all non-abelian simple groups.

### Theorem (Brauer-Fowler, 1956)

*Let $H$ be a finite group. Then there are only finitely many finite simple groups $G$ containing an involution $t$ in $G$ with $C_G(t)$ isomorphic to $H$.*

Figure: Thompson and Tits, winners of the Abel price for their work on finite simple groups

### Theorem

*The finite simple groups are known!*

### Theorem (Schreier's conjecture)

*The outer automorphism group of a simple group is solvable.*

# Towards a proof of Burnside's Theorem

Theorem

*If $G$ is a finite group containing a conjugacy class $C$ of size a power of a prime, then $N = \langle gh^{-1} \mid g, h \in C \rangle$ is a proper normal subgroup of $G$.*

Proof of Burnside's Theorem.

Fix a Sylow $p$-subgroup $P$ of $G$. Then $Z(P)$ is non-trivial. Let $1 \neq z \in Z(P)$.

Now $P \leq C_G(z)$. So, either $z \in Z(G)$ or $z^G$ is a conjugacy class of order $q^n$.

In both cases $G$ contains a proper normal subgroup, and we can apply induction to finish the proof. □