MasterMath: Representation Theory

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Select 4 exercises in total from this and previous sheets to hand in (on paper or electronically to s.h.yu@tue.nl but please send all Leiden exercises to gdt@math.leidenuniv.nl) by Tuesday November 30^{nd} 2010.

1. Consider the group $G = \mathfrak{S}_4$ and the subgroup $H = \langle (123) \rangle \cong C_3$.
   (a) If $\chi_1, \ldots, \chi_5$ are the irreducible characters of G (see e.g., Exercise L.71), work out the restrictions $\mathrm{Res}_H^G \chi_i$, for each $1 \leq i \leq 5$, as sums of the irreducible characters ϕ_1, ϕ_2, ϕ_3 of C_3.
   (b) Calculate the induced characters $\mathrm{Ind}_H^G \phi_j$, for each $1 \leq j \leq 3$, as sums of the irreducible characters χ_i of G.

2. Suppose G is a group. Let χ be a character of G and ϕ a character of a subgroup $H \leq G$. For $x \in G$, define the class function φ_x^G on G by

$$\varphi_x^G(g) = \begin{cases} 1 & \text{if } g \in [x]; \\ 0 & \text{if } g \notin [x]. \end{cases}$$

   φ is the characteristic function of the conjugacy class $[x]$.
   (a) Prove that

$$\langle \chi, \varphi_x^G \rangle_G = \frac{\chi(x)}{|C_G(x)|}.$$

   (b) Use part (a) and Frobenius Reciprocity to show that
   (1) If no element of $[x]$ lies in H, then $(\mathrm{Ind}_H^G \phi)(x) = 0$.
   (2)

$$(\mathrm{Ind}_H^G \phi)(x) = |C_G(x)| \left(\frac{\phi(x_1)}{|C_H(x_1)|} + \cdots + \frac{\phi(x_m)}{|C_H(x_m)|} \right),$$

   where $x_1, \ldots, x_m \in H$ and $\mathrm{Res}_H^G(\varphi_x^G) = \varphi_{x_1}^H + \ldots + \varphi_{x_m}^H$.

3. Let G be a finite permutation group with character π over $\mathbb{C}$, and $\chi_1(G)$ be the trivial character of G over $\mathbb{C}$. Then recall that the number of G-orbits is equal to $\langle \pi, \chi_1(G) \rangle_G$. If G is transitive and H is a point stabilizer in G, then prove that the number of H-orbits equals $\langle \pi, \pi \rangle_G = \frac{1}{|G|} \sum_{x \in G} (\pi(x))^2$.

4. Let H be a subgroup of G, φ a character of H, and let χ be a character of G. Prove that

$$\operatorname{Ind}_H^G \left(\varphi(\operatorname{Res}_H^G \chi)\right) = \left(\operatorname{Ind}_H^G \varphi\right) \chi.$$

[**Hint:** Consider the inner product of each side with an arbitrary irreducible character of G, and use Frobenius Reciprocity.]

5. Let G be a group with an abelian subgroup H such that there exists a series of subgroups $H = N_0 \subset \ldots \subset N_l = G$, where each $N_i \triangleleft N_{i+1}$. Prove that if χ is an irreducible complex character of G, then $\chi(1)$ divides $[G : H]$. [**Hint:** Let G be a group with normal subgroup N, and V an irreducible $\mathbb{C}G$-module such that V_N is the direct sum of s irreducible $\mathbb{C}N$-modules, then s divides $[G : N]$.]

6. Consider $G = PSL_3(2)$, the projective special linear group of degree three over the field $\mathbb{F}_2$ of two elements. ($G \cong SL_3(2) \cong GL_3(2) \cong PSL_2(7)$). The group G is simple and has order 168. It is 2-transitive on the 7 points of the projective plane of order 2. The point stabilizer is the subgroup $H = SL_2(2) \ltimes \mathbb{F}_2^2$, which is the split extension of $SL_2(2) \cong \mathfrak{S}_3$ by $\mathbb{F}_2^2$ and is of order 24. This question aims to construct the character table of G using induction from this subgroup H.

   By studying the action of G on the 7 points of the Fano plane, one can determine the conjugacy classes of G whose orders are given in the following table:

conj class	1	C_2	C_4	C_3	C_7	C_7'
number of elts	1	21	42	56	24	24

   The group H has a normal subgroup $N \cong \mathbb{F}_2^2$ with quotient isomorphic to $\mathfrak{S}_3$. The 24 elements of H can be partitioned as follows: the identity element 1, a subset I_3 consisting of 3 involutions in N, a subset I_6 of 6 involutions not in N, a subset E_8 of 8 elements of order 3, and a subset E_6 of 6 elements of order 4. It is well-known that there are two 1-dimensional irreducible representations of $\mathfrak{S}_3$. Composing the map $H \to H/N$ with these produces the following table for the corresponding characters

	1	I_3	I_6	E_8	E_6
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1

   (a) Using an induction of characters formula applied to the characters in the above table, fill in the following table.

conj class	1	C_2	C_4	C_3	C_7	C_7'
number of elts	1	21	42	56	24	24
$\operatorname{Ind}_H^G \chi_1$						
$\operatorname{Ind}_H^G \chi_2$						

(b) Let φ_1 be the trivial character of G. Consider the characters $\varphi_2 := \mathrm{Ind}_H^G(\chi_1) - \varphi_1$ and $\varphi_3 := \mathrm{Ind}_H^G \chi_2$. Explain why φ_2 and φ_3 are irreducible. Furthermore, determine the values of the remaining three irreducible characters φ_4, φ_5, φ_6, evaluated at 1.

(c) Using the properties and relations on characters you have learnt (e.g., the column orthogonality relations), complete the rest of the character table.

conj class	1	C_2	C_4	C_3	C_7	C_7'
number of elts	1	21	42	56	24	24
φ_1						
φ_2						
φ_3						
φ_4						
φ_5						
φ_6						