MasterMath: Representation Theory

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Select 4 exercises in total from this and previous sheets to hand in (on paper or electronically to s.h.yu@tue.nl but please send all Leiden exercises to gdt@math.leidenuniv.nl) by Tuesday December 7^{th} 2010.

1. Prove that there exists an irreducible representation of A_5 of degree 3 over $\mathbb{C}$. Show that this representation is not induced from a representation of some subgroup of A_5.

2. Let μ be a faithful character of a representation of a group G over $\mathbb{C}$; faithful here means that μ has a trivial kernel.
   Put $\mu(G) := \{\mu(x) | x \in G\}$. Then $|\mu(G)|$ is the number of distinct values μ assumes on G. Prove that each irreducible character ζ of G appears with positive multiplicity in at least one of $\mu^0 (= 1_G)$, $\mu, \mu^2, \ldots, \mu^{t-1}$, where $t = |\mu(G)|$. [**Hint:** You may use the fact that the Vandermonde determinant $\det(\mu^i(x_j))$ is non zero].

3. Let H be a subgroup of G and ψ a character of H over $\mathbb{C}$. Let $x \in G$, and let L be the conjugacy class containing x. Prove the following holds.
$$\left(\operatorname{Ind}_H^G \psi\right)(x) = \frac{|C_G(x)|}{|H|} \sum_{y \in L \cap H} \psi(y).$$

4. Prove that the Frobenius Reciprocity formula characterizes induction of class functions. In other words, show that if ψ and ψ' are class functions of H and G, respectively, satisfying the condition
$$(\psi', \zeta) = (\psi, \operatorname{Res}_H^G \zeta), \text{ for all class functions } \zeta \text{ of } G,$$
   then $\psi' = \operatorname{Ind}_H^G \psi$.

5. Show that no simple group has an irreducible character (over $\mathbb{C}$) of degree 2. [**Hints:** You may use the following facts – **(1)** Recall that the *derived* subgroup G' of G is the subgroup of G which is generated by all elements of the form $g^{-1}h^{-1}gh$, where $g, h \in G$. The number of distinct linear characters of G is equal to $|G/G'|$, and so divides $|G|$; **(2)** Suppose ρ is a representation of G over $\mathbb{C}$. Then $\delta : g \to \det(g\rho)$, where $g \in G$, is a linear character of G.

6. **Theorem:** An irreducible representation V (with corresponding character function χ) must be one and only one of the following:

   (1) Complex: χ is not real-valued; V does not have a G-invariant non-degenerate bilinear form.

   (2) Real: $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, where V_0 is a real vector space on which G acts; V has a G-invariant symmetric non-degenerate bilinear form.

   (3) Quaternionic: χ is real, but V is not real; V has a G-invariant skew-symmetric non-degenerate bilinear form. [Recall: **Definition** – A (complex) representation V is *quaternionic* if it has a G-invariant homomorphism $\varphi : V \to V$ which is conjugate linear and satisfies $\phi^2 = -\mathrm{id}$].

   Let V be an irreducible representation of G over $\mathbb{C}$ and χ be the corresponding character function. Show that

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex;} \\ 1 & \text{if } V \text{ is real;} \\ -1 & \text{if } V \text{ is quaternionic;} \end{cases}$$

   (**Note:** This implies that if the order of G is odd, all nontrivial representations must be complex).

7. (a) Let $n > 2$. Consider the representation of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{R}^2$ given by

$$\rho : k \mapsto \begin{pmatrix} \cos(\frac{2\pi k}{n}) & -\sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{pmatrix}$$

   ρ is irreducible over $\mathbb{R}$. Is it also irreducible over $\mathbb{C}$? Explain why.

   (b) Let V_0 be a real vector space on which G acts irreducibly, and $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the corresponding real representation of G. Show that if V is not irreducible, then it has exactly two irreducible factors, and they are conjugate complex representations of G.