MasterMath: Representation Theory

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Week 2 - September 14^{th} 2010

Select 4 in total out of the following exercises and part 2 of last week's exercises to hand in (on paper or electronically to s.h.yu@tue.nl) by Tuesday September 28^{th} 2010.

1. Compute all the composition series of $\mathbb{Z}_{48}$, $\mathfrak{S}_3 \times \mathbb{Z}_2$ and the dihedral group D_5. Also, show that an infinite abelian group has no composition series. [**Hint:** an infinite abelian group always has a proper normal subgroup].

2. Let G be a group of order p^n, where p is prime. Suppose H is a proper subgroup of G. Prove that $N_G(H)$ does contain H as a proper subgroup.

3. Let G be a group of order $p^m q$, where p, q are distinct primes.

   (a) By Sylow's third theorem, the number of Sylow p–subgroups is q. Prove that, if every pair of Sylow p–subgroups has a trivial intersection, then the number of Sylow q–subgroups is 1, and hence G is not simple.

   (b) Now suppose that there exists two Sylow p–subgroups P_1, P_2 of G with non-trivial intersection. Choose such P_1 and P_2 with the largest intersection possible, and let $N_i = N_{P_i}(P_1 \cap P_2)$ and $J = \langle N_1, N_2 \rangle$. Then prove that J is not a p-group. [**Hint:** You may use the fact that if L is a finite p-group, then $H < N_L(H)$, for any subgroup $H < L$, see the above exercise].

   (c) Thus finish the proof that G is not simple by considering a Sylow q-subgroup of J and showing that P_1 contains the intersection of all the normal subgroups of G containing $P_1 \cap P_2$.

4. Let G be a group with a finite normal subgroup K and P denote a Sylow p-subgroup of K. Show that $KN_G(P) = G$. [**Hint:** Show that G acts transitively on the set of Sylow p-subgroups of K by conjugation].

5. If G is a transitive subgroup of $\mathfrak{S}_n$ (i.e. the action of G on $\{1, \ldots, n\}$, as a subgroup of $\mathfrak{S}_n$, is transitive), show that

$$\sum_{g \in G} |Fix(g)| = |G| \quad \text{and} \quad \sum_{g \in G} |Fix(g)|^2 = m|G|,$$

where m is the number of orbits of the point stabilizers of G.

6. Give an example of a transitive permutation group of infinite degree in which every element has infinitely many fixed points.