MasterMath: Representation Theory

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Select 4 in total out of the following exercises to hand in (on paper or electronically to s.h.yu@tue.nl) by Tuesday October 12$^{\text{th}}$ 2010.

1. (a) Let G be the cyclic group $C_3 = \langle\zeta\rangle$, and let V be the 2-dimensional $\mathbb{C}G$-module with basis v_1, v_2 and the action defined by $\zeta v_1 = v_2$ and $\zeta v_2 = -v_1 - v_2$. Express V as a direct sum of simple $\mathbb{C}G$-submodules.

   (b) Let $G = C_2 \times C_2$. Express the group algebra $\mathbb{R}G$ as a direct sum of 1-dimensional $\mathbb{R}G$-submodules.

   (c) Suppose that G is the infinite group $\{\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$, and let V be the $\mathbb{C}G$-module $\mathbb{C}^2$ equipped with the natural action (matrix multiplication). Show that V is not semisimple. [Note: This shows that Maschke's Theorem fails for infinite groups.]

2. Let k be a field and R the ring $M_n(k)$ of all $n \times n$-matrices over k.

   (a) Prove that the module k^n of column vectors is a simple R-module.

   (b) Prove that $M_n(k)$ is a semisimple ring.

3. Suppose R is a semisimple ring with unit element 1. Suppose $R = \bigoplus_{i=1}^{n} R_i$ is a decomposition of R into its simple submodules.

   Prove that there exist elements $e_i \in R_i$ with $R_i = Re_i$, $e_i^2 = e_i$ and $e_i e_j = e_j e_i = 0$ for $i \neq j$.

4. Let k be a field and R the ring $T_n(k)$ of all $n \times n$-matrices over k with zeros below the diagonal.

   Is the module k^n a semisimple R-module?

5. (Converse of Schur's Lemma). Let M be a semisimple A-module, where A is an algebra over a algebraically closed field k. Show that if $\mathrm{Hom}_A(M, M) = k$ then M is simple, as an A-module. Moreover, give an example demonstrating that this result need not be true when M is not semisimple as an A-module.

6. Let R be a ring. The radical N of R is the intersection of all maximal left ideals of R.

   (a) Show that $NM = 0$ for every simple R-module M.

(b) Show that N is also a right ideal

(c) What is the radical of R/N?

7. Let R be a ring with radical N and a left ideal I.

   Show that the following are equivalent

   (a) $I \subseteq N$;

   (b) For any finitely generated (left) R-module M, IM is a proper submodule of M.

8. Let R be a ring with 1 and M a semisimple R-module which is the sum of finitely many simple R-modules. Prove that M is finitely generated.

9. Let D be a division ring. Prove $A = \text{End}_D(D^n)$ is semisimple as a left A-module and A is semisimple as a left D-module.

10. Let P be a simple ring; i.e., it has no non-trivial proper two-sided ideals. Prove that P is a semisimple ring if and only if P has a minimal left ideal. [Hint: If P is a simple and semisimple ring, then $P \cong \text{End}_D(D^n)$, for some division ring D and positive integer n. This is a direct consequence of Wedderburn's Theorem].

11. Let R be a ring.

    (a) Suppose V is a *semisimple* R-module, $B = \text{End}_R(V)$. Then V is a B-module under $\varphi \cdot v = \varphi(v)$, for any $\varphi \in B$, $v \in V$. Show that, for any $v \in V$ and any $f \in \text{End}_B(V)$, there exists an $r \in R$ such that $f(v) = rv$.

    (b) Now suppose V is a *simple* R-module, $B = \text{End}_R(V)$, $f \in \text{End}_B(V)$, and $v_1, \ldots, v_n \in V$. Prove that there exists an $r \in R$ such that $f(v_i) = rv_i$, for all i.