MasterMath: Representation Theory

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Select 4 exercises in total from this and previous sheets to hand in (on paper or electronically to s.h.yu@tue.nl but please send all Leiden exercises to gdt@math.leidenuniv.nl) by Tuesday November 2^{nd} 2010.

1. Prove that for every finite simple group G, there exists a faithful irreducible $\mathbb{C}G$-module. (Recall that the regular $\mathbb{C}G$-module is faithful).

2. (a) Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G. Use Schur's Lemma to show that ρ is irreducible if and only if every $n \times n$ matrix A which satisfies
$$A\rho(g) = \rho(g)A, \forall g \in G$$
has the form $A = \lambda I_n$, where $\lambda \in \mathbb{C}$ and I_n is the $n \times n$ identity matrix.

   (b) Consider the dihedral group $D_8 = \langle a, b \,|\, a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and the representation ρ of D_8 over $\mathbb{C}$ defined by the following:
$$\rho(a) = \begin{pmatrix} -7 & 10 \\ -5 & 7 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}.$$
Use part (a) of this question to determine whether or not ρ is irreducible.

3. (a) Suppose G is a finite group and there exists a faithful irreducible $\mathbb{C}G$-module. Prove that the center $Z(G)$ is cyclic.

   (b) Does the group $C_2 \times D_8$ have a faithful irreducible representation? If so, give an example.

4. (a) Give all irreducible $\mathbb{C}G$-modules for $G = C_2 \times C_2$. Are any of these faithful?

   (b) Consider the quaternion group $Q_8 = \langle c, d \,|\, c^4 = 1, c^2 = d^2, d^{-1}cd = c^{-1}\rangle$. Does Q_8 have a faithful irreducible representation? If so, give an example.

5. Let G be a finite group. Prove that G is abelian if and only if all irreducible $\mathbb{C}G$-modules have dimension 1. (Again, recall that the regular $\mathbb{C}G$-module is faithful).