MasterMath: Representation Theory

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Select 4 exercises in total from this and previous sheets to hand in (on paper or electronically to s.h.yu@tue.nl but please send all Leiden exercises to gdt@math.leidenuniv.nl) by Tuesday November 9$^{\text{nd}}$ 2010.

1. (a) Describe the restriction of the irreducible representations of $\mathfrak{S}_3$ to $\mathfrak{S}_2$, and give the decomposition of $\mathrm{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_3}(U)$, where U is the sign representation of $\mathfrak{S}_2$, into irreducibles.

   (b) Let G be the dihedral group of order 8 with presentation $D_8 = \langle a, b \,|\, a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ and let H be the subgroup $\langle a^2, b\rangle$. Define V to be the $\mathbb{C}H$-submodule of $\mathbb{C}H$ spanned by $1-a^2+b-a^2b$. Find a basis of the induced $\mathbb{C}G$-module $\mathrm{Ind}_H^G(V)$. Is it irreducible?

2. Let G be a finite group and k an algebraically closed field. Suppose $k[G]$ is semisimple. Use Frobenius reciprocity to prove that every simple submodule S of $k[G]$ occurs with multiplicity $\dim_k(S)$ in $k[G]$.

3. Let G be the symmetric group $\mathfrak{S}_n$, where $n \geq 6$, and X_i be the set of all subsets of $\{1, \ldots, n\}$ of size i.

   (a) Suppose $i,\, j < \frac{n}{2}$. Let H be the stabilizer in G of an element in X_i. Determine the number of orbits of H on X_j.

   (b) Prove that the permutation representation of G on X_i over the field $\mathbb{C}$ is the sum of $i+1$ distinct simple submodules.

4. Let k be a field. Suppose G is a finite group and H a subgroup of G.

   Prove that if W is a $k[H]$-module and V a $k[G]$-module, then $\mathrm{Ind}_H^G(W \otimes_k \mathrm{Res}_H^G(V))$ and $\mathrm{Ind}_H^G(W) \otimes_k V$ are isomorphic as $k[G]$-modules.

5. Let H be a normal subgroup of a finite group G. Suppose V is a $k[G]$-module and W a $k[H]$-submodule of $\mathrm{Res}_H^G(V)$.

   (a) Show that gW is also a $k[H]$-submodule, for any $g \in G$.

   (b) Show that $\mathrm{Ind}_H^G(W) = \mathrm{Ind}_H^G(gW)$, for any $g \in G$.

   (c) Let $N_G(W) = \{g \in G \mid gW \text{ is isomorphic to } W\}$. Prove that $N_G(W)$ is a subgroup of G containing H, and that $|G|/|N_G(W)|$ is the number of non-isomorphic gW.

(d) Prove that $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(W)) = \frac{|N_G(W)|}{|H|} \bigoplus_{g\in R} gW$, where R is a set of left coset representatives of $N(W)$ in G.

6. (Clifford's Theorem) Let H be a normal subgroup of a finite group G and k an algebraically closed field of characteristic 0 or $p \not| |G|$.

   Suppose V is a simple $k[G]$-module and W a simple $k[H]$-submodule of $\mathrm{Res}_H^G(V)$. Recall the definition of $N_G(W)$ from the previous question.

   Let $V_1 = \Sigma_{g\in S}\, gW$, where S is a set of representatives of $N_G(W)/H$ and $V_2 = \Sigma_{g\in R\setminus S}\, gW$, where R is a set of representatives of G/H containing S.

   (a) Prove that $V = V_1 + V_2$.

   (b) Prove that $V_1 \cap V_2 = 0$.

   (c) Prove that $V = \mathrm{Ind}_{N_G(W)}^G(V_1)$.

   (d) Prove that V_1 is a simple $N_G(W)$-module.