Problem #23 (Solved !)

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Summary: Must any termination ordering used for proving termination of the Battle of Hydra and Hercules-system have the Howard ordinal as its order type?

The following system [DJ90], based on the “Battle of Hydra and Hercules” in [KP82], is terminating, but not provably so in Peano Arithmetic:

\[
\begin{align*}
  h(z, e(x)) & \rightarrow h(c(z), d(z, x)) \\
  d(z, g(0, 0)) & \rightarrow e(0) \\
  d(z, g(x, y)) & \rightarrow g(e(x), d(z, y)) \\
  d(c(z), g(g(x, y), 0)) & \rightarrow g(d(c(z), g(x, y)), d(z, g(x, y))) \\
  g(e(x), e(y)) & \rightarrow e(g(x, y))
\end{align*}
\]

Transfinite ($\epsilon_0$-) induction is required for a proof of termination. Must any termination ordering have the Howard ordinal as its order type, as conjectured in [Cic90]?

Remark

If the notion of termination ordering is formalized by using ordinal notations with variables, then a termination proof using such orderings yields a slow growing bound on the lengths of derivations. If the order type is less than the Howard-Bachmann ordinal then, by Girard’s Hierarchy Theorem, the derivation lengths are provably total in Peano Arithmetic. Hence a termination proof for this particular rewrite system for the Hydra game cannot be given by such an ordering [Andreas Weiermann, personal communication].

Remark

This has been answered to the negative by Georg Moser [Mos09], by giving a reduction order that is compatible with the above rewrite system, and whose order type is at most $\epsilon_0$ (the proof theoretic ordinal of Peano arithmetic).
Bibliography


