

# Novel Time-domain method for free-running oscillators

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## Novel Time-domain Methods for Free-running Oscillators

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**Abstract** — A novel time-domain method for finding the periodic steady-state of a free-running electrical oscillator is introduced. The method is based on the extrapolation technique MPE. This method is applied to the well-known Colpitt's Oscillator, for which it turns out to have super-linear convergence.

### 1 Introduction

The prediction of the behaviour of a particular electrical circuit is of importance for the construction and design of efficient electronic devices. This paper focuses on determining the periodic steady-state (PSS) of an oscillator circuit, which is a typical component occurring in microprocessors and radio frequency (RF) applications. In particular, designers are interested in the effects of noise (small random fluctuations) on such circuits.

One may distinguish between non-autonomous and autonomous circuits. Non-autonomous (or driven) circuits are dictated by a time-dependent input signal. A common situation is that the input signal is periodic, and the output signal is periodic with the same period as the input signal. Consequently, the period  $T$  of the output signal is known a priori. On the other hand, autonomous (or free-running) oscillator circuits have no time-dependent input signal, which means that it is in general not possible to predict the period  $T$  a priori. Moreover, free-running oscillators respond in a special way to noise, which is manifested in phase noise (also called timing jitter).

The standard approach to find the amount of noise in a circuit is to split the problem into two parts.

1. Compute the Periodic Steady State (PSS) under the assumption that there is no noise.
2. Linearise around the solution found under 1. Compute the effect of adding noise sources under the assumption that the noise is sufficiently small for this linearisation to be warranted.

For non-autonomous oscillators many efficient solution methods for finding the PSS exist. For an overview, see [10], for a more recent overview see [12] or [6]. However, for autonomous or free-running oscillators, the situation is less satisfactory. Here, the period  $T$  is an additional degree of freedom, which makes the resulting system under-determined. Most methods proposed

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for the autonomous case have been based on enhancing methods for non-autonomous oscillators [1, 5, 7, 8, 13]. Typically, this is done by considering the period  $T$  as an additional degree of freedom. Unfortunately, many of these methods are very sensitive with respect to the initial guess  $T_0$  for the circuit's period  $T$ . They converge only for  $T_0$  in a small neighbourhood of  $T$ , and they require additional damping strategies in practice [7, 8, 13].

Moreover, it turns out that in the case of an autonomous oscillator, we cannot treat the effect of small noise sources as a linear disturbance. This is discussed in detail in [2], which also discusses several methods to deal with this additional problem.

This paper presents a novel method for the solution of autonomous oscillators. The paper is composed as follows. Section 2 introduces the concept of a periodic steady state. Section 3 presents a straightforward Poincaré-map method. Although this method is robust, it converges linearly for most real-world circuits. In particular, it converges linearly for Colpitt's oscillator, which is a well-known benchmark problem in the literature. For that reason, section 4 introduces an accelerated variant of the Poincaré-map method, based on minimal polynomial extrapolation (MPE). This method gives super-linear convergence for Colpitt's oscillator. Conclusions are presented in section 5. For a more detailed discussion of the method, see [4].

### 2 Periodic steady-states

In this paper, we will concentrate on methods for finding a stable periodic steady-state (see def. 1 and 3 below). Periodic steady-states that are not stable are not interesting for the IC designer, since they do not correspond to any physical behaviour of the modelled circuit. In fact, we want to actively avoid non-stable periodic steady-states for this reason.

**Definition 1.** Consider an autonomous differential-algebraic equation (DAE) of the form:

$$\frac{d}{dt} \mathbf{x}(t) + \mathbf{J}(\mathbf{x}(t)) = \mathbf{0}. \quad (1)$$

A function  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n$  is called a periodic steady-state (PSS) of (1) if:

1.  $\mathbf{x}$  is a solution to (1).
2.  $\mathbf{x}$  is periodic, i.e. there is a  $T > 0$  such that for all  $t \in \mathbb{R}$ ,  $\mathbf{x}(t) = \mathbf{x}(t + T)$ .

This recursion is only in terms of the circuit state  $\mathbf{x}$ ; the period  $T$  does not appear explicitly in this iteration. Suppose that the sequence (6) converges linearly to some fixed point  $\mathbf{x}$  of  $F$ . We look for a way to accelerate this to super-linear convergence. An acceleration method operates on the first  $k$  vectors of a sequence  $\{\mathbf{x}_i\}$ , and produces an approximation  $\mathbf{y}$  to the limit  $\mathbf{x}$ . This approximation is then used to restart (6) with  $\mathbf{y}_0 = \mathbf{y}$  and generate a new sequence  $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \dots$ . Again, the acceleration method can be applied to this new sequence, resulting in a new approximation  $\mathbf{z}$  of the limit. The idea is that the sequence  $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$  converges much faster to the limit of  $\{\mathbf{x}_i\}$  than the sequence  $\{\mathbf{x}_i\}$  itself. Typically, if  $\{\mathbf{x}_i\}$  converges linearly, then  $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots\}$  converges super-linearly. A well-known acceleration method is minimal polynomial extrapolation (MPE). Rather than describing MPE here in detail, the reader is referred to [11]. The MPE-accelerated Poincaré-map method has also been tested on Colpitt's Oscillator. The errors after each iteration have been plotted in Figure 6. Note that for the MPE-accelerated Poincaré-map method, only the outer loop iterations have been plotted. In this problem, every outer loop iteration does 3 inner loop iterations.

computed circuit behaviour. A possible recurring feature is the point at which a specific condition is satisfied. This is equivalent to carrying out a Poincaré-map iteration, see [3], section 1.16. The idea is to cut the transient solution  $\mathbf{x}(t)$  by a hyperplane. The hyperplane is defined by an affine equation of the form  $\langle \mathbf{x}(t), \mathbf{n} \rangle = \alpha$ , where  $\mathbf{x}$  is vector and scalar  $\alpha$ . This equation is called the switch equation. The situation is visualised in Figure 4. The unaccelerated Poincaré-map method can now be described as follows.

**Algorithm 1.** Let an approximate solution  $\mathbf{x}_0$  and a required accuracy tolerance  $\varepsilon > 0$  be given. The approximated solution  $\mathbf{x}$  and period  $T$  is computed by:

$i := 0, \mathbf{t}_0 := 0, \mathbf{x}_0 := \text{some initial guess for } \mathbf{x}$   
**repeat**  
  Starting with  $t = t_i$ ,  $\mathbf{x}(t_i) = \mathbf{x}_i$ ,  
  integrate (1) until  $\langle \mathbf{x}(t), \mathbf{n} \rangle = \alpha$  and  
   $d\langle \mathbf{x}(t), \mathbf{n} \rangle / dt > 0$ .  
   $\mathbf{x}_{i+1} := \mathbf{x}(t_i)$ ,  $t_{i+1} := t_i$   
   $\delta := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$   
   $i := i + 1$   
**until**  $\delta \leq \varepsilon$   
 $T := t_i - t_{i-1}$ ,  $\mathbf{x} := \mathbf{x}_i$

This method has been tested on Colpitt's Oscillator. Figure 5 reveals linear convergence.

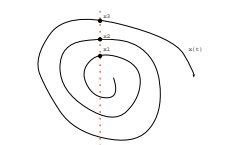


Figure 4: The trajectory of a solution, cut with a hyperplane

### 4 The MPE accelerated Poincaré-map method

Let  $\mathbf{x}(t)$  be the solution of (1) with  $\mathbf{x}(0) = \mathbf{x}_0$ , and  $T_0$  is the smallest  $t > 0$  such that  $\langle \mathbf{x}(t), \mathbf{n} \rangle = \alpha$  and  $d\langle \mathbf{x}(t), \mathbf{n} \rangle / dt > 0$ . Thus  $T_0$  depends on  $\mathbf{x}_0$  as well. Now we can define a function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$F(\mathbf{x}_0) := \mathbf{x}(T_0). \quad (5)$$

The successive approximations of the Poincaré-map method satisfy the recursion relation

$$\mathbf{x}_{i+1} = F(\mathbf{x}_i). \quad (6)$$

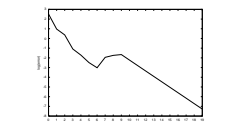


Figure 5:  $\log(\|\mathbf{x}_i - \mathbf{x}_{i-1}\|)$  after each iteration of the Poincaré-map method for Colpitt's Osc.

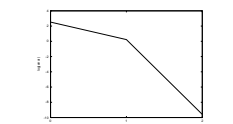


Figure 6:  $\log(\|\mathbf{x}_i - \mathbf{x}_{i-1}\|)$  after each outer-loop iteration of the MPE-accel. Poincaré-map method for Colpitt's Osc.

Note that according to this definition, a stationary solution, i.e. a solution of the form  $\mathbf{x}(t) = \mathbf{x}_0$ , is also a PSS.

**Definition 2.** The limit cycle  $C(\mathbf{x})$  of a PSS  $\mathbf{x}$  is the range of the function  $\mathbf{x}(t)$ , i.e.

$$C(\mathbf{x}) = \{\mathbf{x}(t) | t \in \mathbb{R}\}. \quad (2)$$

A set  $C$  is called a limit cycle of (1) if there is a PSS  $\mathbf{x}$  of (1) so that  $C = C(\mathbf{x})$ .

**Definition 3.** A periodic steady-state  $\mathbf{x}$  is called stable<sup>1</sup> if there is a  $\delta > 0$  so that the following holds: For every solution  $\mathbf{x}^*$  to (1) which has the property that

$$\exists \tau_1, \tau_2 \geq 0 \quad \|\mathbf{x}^*(\tau_1) - \mathbf{x}(\tau_1)\| < \delta, \quad (3)$$

there exists a  $\tau_2 > 0$  so that

$$\lim_{t \rightarrow \infty} \|\mathbf{x}^*(t) - \mathbf{x}(t + \tau_2)\| = 0 \quad (4)$$

A limit cycle is called stable when all of its periodic steady-states are stable.

A well-known example of a free-running oscillator is Colpitt's Oscillator. Its network schematics are shown in figure 1. The transistor is a Philips model known as Bipolar NPN Transistor TN Level 503. A description is available from the Philips Semiconductors web-site [9]. Colpitt's oscillator converges to a PSS after some time. The computed PSS is shown in figure 2 and 3.

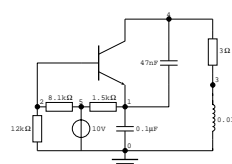


Figure 1: Colpitt's Oscillator

### 3 The Poincaré-map method

The Poincaré-map method for solving (1) is based on the following observation. Provided we start sufficiently close to a stable limit cycle  $C$ , a transient simulation will eventually converge towards  $C$ . After all, this

<sup>1</sup>some authors prefer the term smoothly stable

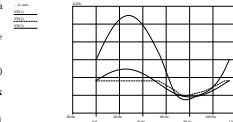


Figure 2: Nodal voltages at the PSS, computed for Colpitt's Oscillator

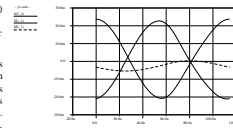


Figure 3: Branch currents at the PSS, computed for Colpitt's Oscillator

implied in the definition of a stable limit cycle. Therefore, we can simply approximate the PSS by starting at some point  $\mathbf{x}_0$  and then performing a transient simulation until the solution  $\mathbf{x}(t)$  has approached the stable limit cycle sufficiently close. There are, however, two problems with this approach

- We have to find a way to detect when we have approached the stable limit cycle close enough, so that we know when to stop.
- Convergence will be linear at best, which means that excessive computing time is needed to arrive at a solution.

This section addresses the first problem. The proposed solution method is still hampered by the second problem; therefore, it will be rather slow. However, in the next section we shall show how we can accelerate the method.

First we note that the length of the period can be estimated by looking for periodic recurring features in the

### 5 Conclusions

The Poincaré-map method and the MPE-accelerated Poincaré-map method have been tested on Colpitt's Oscillator. For the Poincaré-map method, convergence becomes linear after several iterations. The MPE-accelerated method leads to much faster convergence than the unaccelerated method.

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