On linear subspaces of the Hilbert nullcone and polarization in invariant theory

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Consider the usual representation of $\text{SL}_n$ on the symmetric bilinear forms $\text{Sym}_n$ by means of $g \cdot A \mapsto (g^{-1})^t A g^{-1}$. Let $H \subset \text{Sym}_5$ be a subspace on which the determinant vanishes identically.

**Question:** Is $H$ equivalent to a subspace of either

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\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
* & * & * & * \\
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* & * & * & * \\
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\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
* & * & * & * \\
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\end{bmatrix}
\]

under the above operation of $\text{SL}_5$?

This question has the following background in classical invariant theory: For a reductive group $G$ and a complex representation $V$ we denote by $\mathcal{O}(V)^G$ the ring of invariant polynomial functions. The Hilbert nullcone $\mathcal{N}_V \subset V$ is the zero set of all non-constant homogeneous elements of $\mathcal{O}(V)^G$. Even when $V$ is irreducible, finding the generators of $\mathcal{O}(V)^G$ is usually very difficult. Even more so, if we are looking for the generators of $\mathcal{O}(V^\oplus k)^G$, where the operation of $G$ on $V^\oplus k$ is given by $g(v_1, \ldots, v_k) = (gv_1, \ldots, gv_k)$. In this talk I will explain an interesting connection between the structure of the linear subspaces of the nullcone $\mathcal{N}_V$ on one hand, and the question, whether
A certain set of invariants of $O(V^\oplus k)^G$ (obtained by the classically known *polarization process*) defines the nullcone $N_{V^\oplus k} \subset V^\oplus k$ on the other hand.

By a result of HILBERT, finding invariants that define the nullcone $N_{V^\oplus k} \subset V^\oplus k$ is an important step in finding a complete set of generators for $O(V^\oplus k)^G$.

For the representation of $SL_n$ on $Sym_n$ the invariant ring is generated by the determinant and hence the nullcone $N_{Sym_n}$ is the set of all forms on which the determinant vanishes. For $n = 5$ the above mentioned connection leads exactly to the question posed above. Its answer is ‘no’, however.