IMPROVED CORRELATION RECEIVER FOR FRAME SYNCHRONIZATION

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We study a modification of the classical correlation receiver for frame synchronization. The modification consists in introducing a slicer before the correlator. It turns out that apart from having a lower complexity, the modified receiver outperforms the classical receiver for practical values of the probability of synchronization error.

DEFINITIONS AND SYSTEM MODEL

We will use bold letters to denote a sequence $x$ of length $L$ and regular letters to denote its elements $x[n]$ for $n = 0, \ldots, L - 1$ (we start indexing at zero).

We assume a communication system where the transmitted sequence is built up as a succession of frames. We define each frame $f$ to consist of three parts: a preamble $p$, a sync-word $s$ and data $d$ (in this order) of respective lengths $N_p$, $N_s$ and $N_d$. The frame can alternatively be specified as $f = (p, s, d)$ and has a length $N = N_p + N_s + N_d$. We consider $f \in \{-1, 1\}^N$.

The preamble is usually a periodic sequence used for bit synchronization. The sync-word is a short sequence (typically not longer than 20 bits) used to mark the beginning of the data part.

Error Probabilities

The task of the frame synchronization is to identify the beginning of the data in a frame. This is done by signalling the end of the sync-word. We introduce the End of Sync-word Instant (ESI), which is the instant at which detection should ideally occur.

$End$ of Sync-word Instant $i_d$: The end of sync-word instant $i_d$ is the position of the last bit of the sync-word in the frame i.e. $i_d \doteq N_p + N_s - 1$.

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When designing the frame synchronization, an important measure of performance is the achieved \textit{Probability of Synchronization Error} $P_{SE}$. Before defining $P_{SE}$ we identify the causes for synchronization errors. Two frame synchronization error events can be identified: the \textit{false alarms} and the \textit{missed detection}.

\textit{Probability of False Alarm at instant $i < i_d$} $P_{FA}[i]$: The probability of false alarm at instant $i < i_d$, $P_{FA}[i]$, is the probability that the sync-word is declared as found at an instant $i$ prior to the ESI $i_d$.

\textit{Probability of Missed Detection} $P_{MD}$: The probability of missed detection $P_{MD}$ is the probability that the sync-word is not detected at the ESI $i_d$. \textit{Probability of Synchronization Error} $P_{SE}$: The probability of synchronization error $P_{SE}$ is the probability that frame synchronization fails because either a false alarm occurred at any instant before $i_d$ or the detection of the sync-word was missed.

Therefore, assuming that the frame synchronization receiver starts searching for the sync-word at the beginning of the preamble\textsuperscript{1}, we write

\begin{equation}
P_{SE} = 1 - (1 - P_{MD}) \cdot \prod_{i=0}^{i_d-1} (1 - P_{FA}[i]). \tag{1}
\end{equation}

In order to find a simple expression for $P_{SE}$ we assume that the $P_{FA}$ is only non-negligible at one unique instant (this assumption will be true for a well designed sync-word). Therefore we define $P_{FA}^{\max} = \max_{i < i_d} P_{FA}[i]$ and approximate expression (1) by

\begin{equation}
P_{SE} \geq 1 - (1 - P_{MD}) \cdot (1 - P_{FA}^{\max}) \approx P_{MD} + P_{FA}^{\max}. \tag{2}
\end{equation}

\textbf{Correlation Receiver}

Barker \textsuperscript{1} proposed the correlation receiver in his pioneering work on frame synchronization. The correlation receiver consists of a correlator and a threshold detector (see Fig. 1). The correlation receiver searches for the sync-word by correlating the received sequence with the sync-word. It flags detection whenever the correlation sequence surpasses a given threshold.

\textsuperscript{1}Note that in practice the receiver starts searching for the sync-word at an unknown position of the preamble. However, for the sake of simplicity, we assume that it starts at the beginning.
Figure 1: Block diagram of the frame synchronization system model.

**Correlator**

The correlator generates $q_w$, the correlation sequence of the received sequence $r$ with a stored sequence $w$ denoted as the detection-word. We will consider $w = s$ which is the usual choice in the literature, however it is straightforward to adapt our results to the case where $w \neq s$.\(^2\) We can write

$$q_w[i] = \sum_{k=0}^{N_s-1} w[k] \cdot r[i + k - (N_s - 1)] \quad \text{for } i = 0, \ldots, N - 1. \quad (3)$$

Since $r = f + n$, we can write $q_w[i] = c_w[i] + n_w[i]$ for $i = 0, \ldots, N - 1$ where $c_w$ is the correlation sequence of frame $f$ with $w$ i.e.

$$c_w[i] = \sum_{k=0}^{N_s-1} w[k] \cdot f[i + k - (N_s - 1)] \quad \text{for } i = 0, \ldots, N - 1 \quad (4)$$

and $n_w$ is the correlation sequence of the noise sequence $n$ with $w$. We consider the elements of $n$ to be statistically independent Gaussian random variables with 0 mean and variance $\sigma^2$. Therefore $n_w$ is additive Gaussian noise of variance $\sigma_w^2 = ||w||^2 \cdot \sigma^2 = N_s \cdot \sigma^2$ at the output of the correlator, with $||w||^2 = \sum_{i=0}^{N_s-1} w^2[i] = N_s$.

**Threshold Detector**

The threshold detector flags detection whenever the output sequence of the correlator $q_w = c_w + n_w$ surpasses a given threshold which we now proceed to design. We require detection to occur at the ESI $i_d$, therefore we need that

\(^2\)In [2] the advantages of considering $w \neq s$ are investigated.
\( c_{\text{w}}[i_d] > \max \{c_{\text{w}}[i] : i < i_d\} \) and choose a threshold \( T \) satisfying \( c_{\text{w}}[i_d] > T > \max \{c_{\text{w}}[i] : i < i_d\} \). The presence of the additive Gaussian noise \( n_{\text{w}} \) results in a \( P_{\text{MD}} \) of

\[
P_{\text{MD}} = Q \left( \frac{c_{\text{w}}[i_d] - T}{\| \text{w} \| \cdot \sigma} \right)
\]

and a \( P_{\text{FA}}[i] \) of

\[
P_{\text{FA}}[i] = Q \left( \frac{T - c_{\text{w}}[i]}{\| \text{w} \| \cdot \sigma} \right) \quad \text{for } i < i_d
\]

where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-r^2/2} dr \). From expressions (5) and (6) we see that choosing \( T \) to decrease either of the probabilities increases the other. Since both error events result in a frame synchronization error, we choose \( T \) in order to minimize the maximum of \( P_{\text{MD}} \) and \( P_{\text{FA}}[i] \). The optimal threshold \( T_{\text{opt}} \) in the above sense is the one yielding \( P_{\text{MD}} = P_{\text{FA}}^{\text{max}} \) where

\[
P_{\text{FA}}^{\text{max}} = \max_{i < i_d} P_{\text{FA}}[i] = Q \left( \frac{T - \max \{c_{\text{w}}[i] : i < i_d\}}{\| \text{w} \| \cdot \sigma} \right).
\]

This results in

\[
T_{\text{opt}} = (c_{\text{w}}[i_d] + \max \{c_{\text{w}}[i] : i < i_d\}) / 2
\]

and

\[
P_{\text{MD}} = P_{\text{FA}}^{\text{max}} = Q \left( \frac{(c_{\text{w}}[i_d] - \max \{c_{\text{w}}[i] : i < i_d\})/2}{\| \text{w} \| \cdot \sigma} \right) = Q \left( \frac{D/2}{\| \text{w} \| \cdot \sigma} \right)
\]

where \( D \) is the detection distance, which we define next.

**Detection Distance** \( D \): The detection distance is the difference of the value of \( c_{\text{w}} \) at the ESI and the maximum value at any prior instant i.e. \( D = c_{\text{w}}[i_d] - \max \{c_{\text{w}}[i] : i < i_d\} \).

Note that \( D \) is a function of the preamble \( p \) and the sync-word \( s \). We now use (9) in (2) and obtain an approximate expression for the probability of synchronization error

\[
P_{\text{SE}} \approx 2 \cdot Q \left( \frac{D/2}{\| \text{w} \| \cdot \sigma} \right).
\]

**IMPROVED CORRELATION RECEIVER**

The receiver depicted in Fig. 2 converts the received sequence \( r \) into binary by means of a slicer. This implies that the correlation can be computed by
adding binary values solely and therefore the correlation receiver has a very low complexity. We denote this correlation receiver as the binary correlation receiver. The slicer converts the real valued sequence \( r = f + n \) into the binary sequence \( b \) (see Fig. 2). Therefore we write for \( b \)

\[
\begin{align*}
b[i] &= \begin{cases} 
+1 & \text{if } r[i] \geq 0 \\
-1 & \text{if } r[i] < 0 
\end{cases} 
\quad \text{for } i = 0, \ldots, N - 1 
\end{align*}
\]

i.e. positive values are detected as 1’s and negative values as -1’s.\(^3\)

We can now write \( b = f + e \), where \( e \in \{-2, 0, 2\}^N \) is the error sequence. A value \( e[i_0] = -2 \) (resp. 2) at an instant \( i_0 \) signals that bit \( f[i_0] = 1 \) (resp. -1) was detected erroneously as \( b[i_0] = -1 \) (resp. 1).\(^4\) The bit error probability (bEP) \( \beta \) i.e. the probability that a bit is detected erroneously due to the AWGN (of mean 0 and variance \( \sigma^2 \)) is \( \beta = P(e[i] \neq 0) = Q(1/\sigma) \) for \( i = 0, \ldots, N - 1 \).

The binary sequence \( b \) enters the correlator. At the output of the correlator we will have the correlation sequence \( q_w = c_w + e_{N_s} \) where \( e_{N_s} \in \{-2N_s, \ldots, -2, 0, 2, \ldots, 2N_s\}^N \) is the correlation sequence of \( e \) with \( w \in \{-1, 1\}^{N_s} \) of length \( N_s \). (Note that the only assumption we make for \( w \) is to be binary.) Note that now the noise sequence takes only discrete values. We derive general expressions for \( P_{MD} \) and \( P_{FA} \) in the next section.

**ERROR PROBABILITIES FOR BINARY CORRELATION RECEIVER**

The following definition will prove useful to express the probability distribution of \( e_{N_s} \).

\(^3\)We could have chosen \{0, 1\} or any other binary alphabet for \( b \).

\(^4\)Note that \( e[i_0] = -2 \) (resp. 2) is not allowed when \( f[i_0] = -1 \) (resp. 1), since \( b[i_0] \in \{-1, 1\} \).
Probability distribution $P_{q}(L, l, j)$

Given a binary sequence of length $L$ and a bit error probability $q$, we compute $P_{q}(L, l, j)$ the probability of having $j$ errors more in any subsequence of $l$ bits than in the remaining $L - l$ bits. We write

$$P_{q}(L, l, j) = \min (l-j, L-l) \cdot \sum_{i=0}^{\min (l-j, L-l)} \binom{L-l}{i} \cdot q^i \cdot (1-q)^{(L-l)-i} \cdot \binom{l}{i+j} \cdot q^{i+j} \cdot (1-q)^{l-(i+j)} \quad (12)$$

where the first binomial term corresponds to having $i$ errors in the subsequence of $L - l$ bits and the second binomial term corresponds to having $i + j$ errors in the subsequence of $l$ bits.

Probability distribution of $e_{N_s}$

The probability distribution for each element of $e_{N_s} \in \{-2N_s, \ldots, -2, 0, 2, \ldots, 2N_s\}^N$ can be written as

$$P(e_{N_s}[i] = -2j) = \begin{cases} P_{\beta}(N_s, l[i], j) & \text{for } j = 0, \ldots, l[i] \\ 0 & \text{for } l[i] < j \leq N_s \end{cases}$$

$$P(e_{N_s}[i] = 2j) = \begin{cases} P_{\beta}(N_s, N_s - l[i], j) & \text{for } j = 0, \ldots, (N_s - l[i]) \\ 0 & \text{for } (N_s - l[i]) < j \leq N_s \end{cases} \quad (13)$$

where $l[i]$ is the number of bits that coincide between the detection-word $w$ and $(f[i - (N_s - 1)], \ldots, f[i])$ (the subsequence of the frame with which $w$ is correlated at instant $i$) and $\beta$ is the bEP. The above expression states that a bit of the frame that coincides with the corresponding bit of the detection-word will contribute with $-2$ to $e_{N_s}$ whenever it flips due to an error and a bit that does not coincide will contribute with $+2$ whenever it flips.

Now we can express $P_{MD}$ and $P_{FA}[i]$ in function of $P(e_{N_s}[i])$. We write

$$P_{MD} = P(e_{N_s}[i_d] < -(c_{w,i_d} - T)) = \sum_{j=\lceil(c_{w,i_d} - T + 1)/2\rceil}^{l[i_d]} P(e_{N_s}[i_d] = -2j) \quad (14)$$

$$P_{FA}[i] = P(e_{N_s}[i] > (T - c_{w}[i])) = \sum_{j=\lceil(T - c_{w}[i] + 1)/2\rceil}^{N_s - l[i]} P(e_{N_s}[i] = 2j) \quad \text{for } i < i_d \quad (15)$$
where \( \lceil \cdot \rceil \) is the ceil function and \( T \in \mathbb{N} \) is the threshold level.

For \( \mathbf{w} = \mathbf{s} \) we have that \( l[i_d] = N_s \) in expression (13). Furthermore we assume that the detector threshold is given by (8) which implies that \( c_w[i_d] = T_{opt} = D/2 \), where \( D \) is the detection distance. This enables us to compute \( P_{MD} \) and approximate it by

\[
P_{MD} \approx \left( \left\lfloor \frac{N_s}{D+2} \right\rfloor \right) \cdot Q \left( \frac{1}{\sigma} \right)^{\left\lfloor \frac{D+2}{4} \right\rfloor} \quad \text{(with slicer)}.
\]

**CORRELATION RECEIVER PERFORMANCE: BINARY VS. CLASSICAL**

Now we will compare the performance of the binary correlation receiver (with slicer) with the classical receiver (without slicer). Since \( P_{SE} \) is approximately proportional to \( P_{MD} \) (see (10)), we base our comparison on \( P_{MD} \) in order to facilitate the analysis. Comparing (16) with (9) we see that the introduction of the slicer results in an asymptotic gain (in dB) given by

\[
G = 10 \log_{10} \frac{4N_s}{D^2} \left\lfloor \frac{D + 2}{4} \right\rfloor.
\]

We have plotted the exact expressions of \( P_{MD} \) for both receivers in Fig. 3 for a sync-word with \( N_s = 13 \) and \( D = 12 \). The plot also includes a curve corresponding to the \( P_{MD} \) of the ML receiver. The x-axis represents the SNR (signal-to-noise ratio) in dB which is defined as \( \text{SNR} = -20 \log_{10} \sigma \). We see that, for this particular case, at high SNR values (corresponding to practical values of
$P_{MD}$) the receiver with slicer outperforms the classical receiver by about $G = 1.60$ dB.

The reason for the good performance of the binary correlation receiver is that at high SNRs the slicer works as a simple bit detector eliminating much of the noise before the correlator, whereas in the system without slicer the noise values are added by the correlator. However, we expect the classical receiver to be better at low SNRs where the slicer is a poor bit detector and will effectively increase the noise (introducing bit errors) at the input of the correlator. We can observe this behavior in Fig. 4 which shows the low SNR region in more detail. This plot also includes simulation results of both receivers, which match the predicted behavior. We see that, for this example, the slicer starts to pay-off at an SNR of about 4.3 dB.

We conclude that the binary correlation receiver has not only a lower complexity but also has a gain of $G$ dB (see (17)) with respect to the classical correlation receiver for values of $P_{MD}$ of practical interest (e.g. $P_{MD} < 10^{-10}$).

REFERENCES
