

Zeta function of a curve

1 Example

Consider the curve C with equation $X^3Y + Y^3Z + Z^3X = 0$ defined over $k = \mathbf{F}_2$. Let N_d be the number of points with coordinates in \mathbf{F}_{2^d} .

We have $N_1 = 3$: there are three points defined over $\mathbf{F}_2 = \{0, 1\}$, namely $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$.

We have $N_2 = 5$: there are five points defined over $\mathbf{F}_4 = \{0, 1, \omega, \omega^2\}$, namely three over \mathbf{F}_2 and the two points $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$.

We have $N_3 = 24$: there are 24 points defined over $\mathbf{F}_8 = \{0, 1, \zeta, \dots, \zeta^6\}$ where $\zeta^7 = 1$, namely three over \mathbf{F}_2 and the 21 points $(1, \zeta^i, \zeta^{-2i}\alpha)$ where $0 \leq i \leq 6$ and $\alpha^3 + \alpha + 1 = 0$.

Continuing, we find

d	1	2	3	4	5	6	...	9
N_d	3	5	24	17	33	38	...	528

and more generally: $N_d = 2^d + 1$ if $3 \nmid d$, and $N_d = 2^d + 1 - a_d$ if $3 \mid d$, where the a_i are found from $a_3 = -15$, $a_6 = 27$, $a_{3k+6} + 5a_{3k+3} + 8a_{3k} = 0$ ($k \geq 1$).

Put $Z(C, t) = \exp(\sum \frac{1}{i} N_i t^i)$. Then in this example

$$Z(C, t) = \frac{1 + 5t^3 + 8t^6}{(1-t)(1-2t)},$$

a simple rational function that encodes the values of all N_i .

A simpler example is the projective line L . Over \mathbf{F}_q there are $N = q + 1$ points. Now $Z(L, t) = \exp(\sum \frac{1}{i} (q^i + 1)t^i)$. But $\sum \frac{1}{i} t^i = -\log(1-t)$ (for $|t| < 1$), and $\sum \frac{1}{i} q^i t^i = -\log(1-qt)$ (for $|qt| < 1$), so $Z(L, t) = 1/(1-t)(1-qt)$.

Comparing this with the previous we see that a zeta function $Z(C, t) = (1 + 5t^3 + 8t^6)/(1-t)(1-2t)$ corresponds to $N_i = q^i + 1$ when $3 \nmid i$. The recurrence $a_{3k+6} + 5a_{3k+3} + 8a_{3k} = 0$ has solution $a_{3k} = c_1\alpha^k + c_2\beta^k$ if α, β are the two solutions of $x^2 + 5x + 8 = 0$. From $a_0 = 6$, $a_3 = -15$, we see $c_1 = c_2 = 3$. Now $-3 \sum \frac{1}{3i} \alpha^i t^{3i} = \log(1-\alpha t^3)$ so $Z(C, t) = (1-\alpha t^3)(1-\beta t^3)/(1-t)(1-2t) = (1 + 5t^3 + 8t^6)/(1-t)(1-2t)$. In other words, the given expression for $Z(C, t)$ is equivalent to the given values of N_i .

2 Zeta function

Let X be an absolutely irreducible algebraic curve defined over \mathbf{F}_q with N_i points over \mathbf{F}_{q^i} . The zeta function of X is defined as $Z(X, t) := \exp(\sum \frac{1}{i} N_i t^i)$.

Hasse (for $g = 1$) and Weil (for the general case) showed that this function is a rational function of the form $P(t)/(1-t)(1-qt)$ where $P(t)$ is a polynomial

in t . The degree of $P(t)$ is $2g$, where g is the genus of X . The polynomial $P(t)$ has the factorization $P(t) = \prod_{i=1}^{2g} (1 - \alpha_i t)$ where $|\alpha_i| = \sqrt{q}$ for all i .

Taking logarithms we find

$$N_i = 1 + q^i - \sum \alpha^i$$

(where α^{-1} runs through the $2g$ roots of $P(t)$). It follows that

$$|N_1 - (q + 1)| \leq 2g\sqrt{q}.$$

This Hasse-Weil bound was improved by Serre to

$$|N_1 - (q + 1)| \leq g[2\sqrt{q}].$$

(Proof: The α 's are algebraic integers and occur in complex conjugate pairs; if a runs through the g sums $\alpha + \bar{\alpha}$, then for both choices of the sign the product $\prod([2\sqrt{q}] + 1 \pm a)$ is a positive integer, hence at least 1; by the arithmetic-geometric mean inequality we have $\sum([2\sqrt{q}] + 1 \pm a) \geq g$. \square)

Ihara's bound is better for $g > (q - \sqrt{q})/2$:

$$N_1 \leq q + 1 - \frac{1}{2}g + \sqrt{2(q + \frac{1}{8})g^2 + (q^2 - q)g}.$$

(Proof: We have $1 + q - \sum \alpha = N_1 \leq N_2 = 1 + q^2 - \sum \alpha^2$. The α 's occur in complex conjugate pairs with product q , and if a runs through the g sums $\alpha + \bar{\alpha}$ then $1 + q - \sum a \leq 1 + q^2 + 2qg - \sum a^2$. Now use $g \sum a^2 \geq (\sum a)^2$. \square)

If $g = (q - \sqrt{q})/2$ then both Hasse-Weil and Ihara say $N_1 \leq q\sqrt{q} + 1$, and this upper bound is achieved (when q is a square) by the Hermitean curves $X^{r+1} + Y^{r+1} + Z^{r+1} = 0$ where $q = r^2$ (and by no other curves).