Elliptic functions, integrals, and curves

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Abstract

Some elliptic stuff motivated by the current EIDMA course.

1 Elliptic functions

1.1 Definition

Let ω_1, ω_2 be two nonzero complex numbers with non-real ratio. An *elliptic function* (with *periods* $2\omega_1, 2\omega_2$) is a meromorphic function (analytic, with no other singularities than ordinary poles) on the set $\mathbf{C}$ of complex numbers, that is doubly periodic with the given periods, i.e., that satisfies $f(z+2\omega_1) = f(z+2\omega_2) = f(z)$ for all z for which $f(z)$ is defined.

Such a function may be regarded as a function on the quotient $\mathbf{C}/L$, where L is the lattice generated by $2\omega_1, 2\omega_2$, and this is what we mean when we talk about the number of zeros or poles of such a function.

1.2 Order

The elliptic functions with given periods form a field F. The derivative of an elliptic function is again elliptic. Since the poles are isolated and $\mathbf{C}/L$ is compact an elliptic function has only finitely many poles. The double periodicity implies that a contour integral $\int_C f(z)dz$ around a period parallelogram is zero, so that the sum of the residues at the poles is zero. Also, if f is nonzero, $\int_C \frac{f'(z)}{f(z)}dz = 0$ so that f has as many zeros as poles. Thus, if f is nonconstant it has an *order* m: each value $c \in \mathbf{C} \cup \{\infty\}$ is taken precisely m times. And $m > 1$ since a single simple pole cannot have residue 0.

1.3 Location of zeros and poles

Let f be an elliptic function. The sum of the zeros of f equals the sum of the poles (in $\mathbf{C}/L$). Indeed, let S be the sum of the zeros minus the sum of the poles. Then $S = \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz$. On opposite sides of the parallelogram the values for $f(z)$ and $f'(z)$ are equal, but the value of z differs by $2\omega_1$ or $2\omega_2$. But along a side of the parallelogram $\int \frac{f'(z)}{f(z)}dz = [\log f(z)]_a^{a+2\omega}$ is an integral multiple of $2\pi i$, so that $S = 2m\omega_1 + 2n\omega_2 \in L$.

1.4 Existence

One may construct a nonconstant elliptic function $\wp$ (the Weierstrass p function) by putting

$$\wp(z) = \frac{1}{z^2} + {\sum_{m,n}}' \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right),$$

where the sum is over all pairs of integers m, n not both zero.

This function is well-defined (the series converges absolutely and uniformly for z not close to a pole) and is an elliptic function of order 2 with a double pole at 0. The function $\wp(z)$ is even, and its derivative $\wp'(z)$ is odd, of order 3 (with a triple pole at 0 and zeros at ω_1, ω_2 and $\omega_1 + \omega_2$).

Every even elliptic function with periods ω_1, ω_2 is a rational function of $\wp$. An arbitrary elliptic function with these periods is the sum of an even one and $\wp'$ times an even one, so is expressible in terms of $\wp$ and $\wp'$.

1.5 Differential equation

We have

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

for suitable constants g_2, g_3. Indeed, put $\Omega_{m,n} := 2m\omega_1 + 2n\omega_2$ so that $\wp(z) = z^{-2} + \sum'_{m,n}((z - \Omega_{m,n})^{-2} - \Omega_{m,n}^{-2})$. Put $g_2 := 60\sum'_{m,n}\Omega_{m,n}^{-4}$ and $g_3 := 140\sum'_{m,n}\Omega_{m,n}^{-6}$. Then $\wp(z) = z^{-2} + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6)$ for small z, and we see that $\wp'^2 - 4\wp^3 + g_2\wp + g_3 = O(z^2)$ so that this function is analytic near 0, and hence has no poles, hence is zero.

1.6 Addition formula

If u, v, w are nonzero with $u + v + w = 0$, then $\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0$.

Indeed, choose constants a, b so that $\wp'(u) - a\wp(u) - b = \wp'(v) - a\wp(v) - b = 0$. Then the elliptic function $\wp'(z) - a\wp(z) - b$ of order 3 has a triple pole at 0, so the sum of the zeros is also 0. Since u and v are zeros, w also is.

(Note that $\wp(u) \neq \wp(v)$ for $u \neq \pm v$ since $\wp$ is even of order 2.)

2 Elliptic integrals

2.1 Arc length of an ellipse

Finding the length of an arc given by the equation $y = f(x)$ is done by computing the integral $\int \sqrt{(dx)^2 + (dy)^2}$, that is, $\int \sqrt{1 + f'(x)^2} dx$. For an ellipse, given by, say, $x^2 + (y/b)^2 = 1$ (with $b^2 < 1$) this means that we want to compute

$$\int \frac{\sqrt{1 - k^2x^2}}{\sqrt{1 - x^2}}\, dx$$

(with $k^2 = 1 - b^2$). This is an elliptic integral of the second kind.

2.2 Definition

An *elliptic integral* is an integral of the form

$$\int \frac{R(x)}{\sqrt{Q(x)}}\, dx$$

where $R(x)$ is a rational function of x and $Q(x)$ is a cubic or quartic polynomial with real coefficients and positive values in the interval considered.

2.3 Legendre normal form

Since a quartic factors over $\mathbf{R}$ into two quadratic factors we can use a substitution of $(p + qx)/(1 + x)$ for x to get the form

$$\int \frac{R(x)}{\sqrt{\pm(1 \pm mx^2)(1 \pm nx^2)}}\, dx$$

with positive m, n.

If $R(x)$ is an odd function of x, substitute $\sqrt{x}$ for x to get something that is integrable using sin and log. Since an arbitrary $R(x)$ is the sum of an even and an odd part, we may assume that $R(x)$ is an even function of x. Now a substution of $(a + bx^2)/(c + dx^2)$ for x^2 yields the form

$$\int \frac{R(x)}{\sqrt{(1 - x^2)(1 - k^2x^2)}}\, dx$$

with $0 < k < 1$. Possibly after the substitution of $1/kx$ for x we may assume that x^2 lies between 0 and 1.

Split $R(x)$ into fractions and integrate by parts to find that it suffices to consider the three integrals

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \ , \ \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx \ , \ \int \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

where n is real or imaginary. These are called elliptic integrals of the first, second and third kind, respectively.

2.4 Transforming a quartic into a cubic

If a is a zero of $Q(x)$, then the substitution of $x+a$ for x makes the constant term of $Q(x)$ disappear. Now substitution of $1/x$ for x replaces the quartic $Q(x)$ by a cubic $C(x)$. Another substitution of $x+b$ for x makes the coefficient of x^2 in $C(x)$ disappear. Thus, the integral

$$\int \frac{dx}{\sqrt{Q(x)}}$$

is turned into

$$\int \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$$

and the substitution of $\wp(z)$ for x gives us $\int dz$. Thus, elliptic integrals of the first kind are solved by the inverse of the Weierstrass $\wp$ function. Elliptic integrals of the other two kinds are solved using related functions.

(A detail: can one prescribe g_2 and g_3 and find suitable periods ω_1, ω_2? The answer is yes in case the determinant ${g_2}^3 - 27{g_3}^2$ is nonzero.)