Counting symmetric nilpotent matrices

Andries E. Brouwer
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The number of nilpotent matrices is $q^{n(n-1)}$.

$N$ is nilpotent when $N^e = 0$ for some $e \geq 0$.

$e$ is called the exponent of $N$. 
Counting

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The number of nilpotent matrices is $q^{n(n-1)}$.

How many symmetric nilpotent matrices?
Count symmetric nilpotent matrices of order $n$:

$n = 0$: 1 (exponent 0), namely ( )
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$n = 2$:

Look at \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \).

All eigenvalues are 0, so trace is 0, so $c = -a$.

Determinant is 0, so $a^2 + b^2 = 0$.

How many solutions?

$q$ even: $q$

$q \equiv 1 \pmod{4}$: $1 + 2(q - 1) = 2q - 1$

$q \equiv 3 \pmod{4}$: 1

Messy
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$n = 2$: 

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$q \equiv 1 \pmod{4}$: $1 + 2(q - 1) = 2q - 1$

$q \equiv 3 \pmod{4}$: 1

$n = 3$: $1 + (q^2 - 1) + (q^3 - q) = q^3 + q^2 - q$

Exercise

Sometimes we find a polynomial in $q$. 
A matrix $N$ defines a linear map $N : V \to V$ and it makes sense to talk about $N^e$. What does it mean that $N = N^\top$?
Let $g : V \times V \to F$ be a nondegenerate symmetric bilinear form. $N$ is called *self-adjoint w.r.t.* $g$ when $g(x, Ny) = g(Nx, y)$ for all $x, y$. 
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Fix a basis. Then \( g(x, y) = x^\top G y \) for a symmetric matrix \( G \). Now \( N \) is self-adjoint when \( GN = N^\top G \), that is, when \( GN = (GN)^\top \).
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Fix a basis. Then $g(x, y) = x^\top G y$ for a symmetric matrix $G$. Now $N$ is self-adjoint when $GN = N^\top G$, that is, when $GN = (GN)^\top$.

The *standard form* is the one given by the identity matrix: $g(x, y) = x^\top y = \sum x_i y_i$. 
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The standard form is the one given by the identity matrix: $g(x, y) = x^\top y = \sum x_i y_i$.

$N = N^\top$ iff $N$ is self-adjoint for the standard form.
Symmetric bilinear forms

So, it seems we should be counting self-adjoint matrices w.r.t. a given nondegenerate symmetric bilinear form. How many nonequivalent forms are there?
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When \( n \) is even, there are two nonequivalent types.
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When $n$ is odd, all forms are equivalent to the standard form.

When $n$ is even, there are two nonequivalent types. (Assuming $n > 0$.)
So, it seems we should be counting self-adjoint matrices w.r.t. a given nondegenerate symmetric bilinear form. How many nonequivalent forms are there? That depends on the parity of $n$.

When $n$ is odd, all forms are equivalent to the standard form.

When $n$ is even, there are two nonequivalent types.

$q$ odd: the elliptic and hyperbolic forms.

$q$ even: the standard and symplectic forms.
When $n$ and $q$ are even, one has the standard and symplectic forms. How can one distinguish them?
When $n$ and $q$ are even, one has the standard and symplectic forms.

A form $g$ is \textit{symplectic} iff $g(x, x) = 0$ for all $x$. 
When $n$ and $q$ are even, one has the standard and symplectic forms.

A form $g$ is *symplectic* iff $g(x, x) = 0$ for all $x$. That is, iff $G$ has zero diagonal.
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(For $n = 0$ the standard form is symplectic.)
The standard form

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A form \( g \) is *symplectic* iff \( g(x, x) = 0 \) for all \( x \).

When \( n \) is even and \( q \) is odd, one has the elliptic and hyperbolic forms. The form is hyperbolic when \((-1)^{n/2} \det G\) is a square.
The standard form

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A form $g$ is symplectic iff $g(x, x) = 0$ for all $x$.

When $n$ is even and $q$ is odd, one has the elliptic and hyperbolic forms. The form is hyperbolic when $(-1)^{n/2} \det G$ is a square.

The standard form is hyperbolic if $(-1)^{n/2}$ is a square, and elliptic otherwise.
When \( n \) and \( q \) are even, one has the standard and symplectic forms.

A form \( g \) is *symplectic* iff \( g(x, x) = 0 \) for all \( x \).

When \( n \) is even and \( q \) is odd, one has the elliptic and hyperbolic forms. The form is hyperbolic when \((-1)^{n/2} \det G\) is a square.

The standard form is hyperbolic if \((-1)^{n/2}\) is a square, and elliptic otherwise. (For \( n = 0 \) there is no elliptic form.)
The standard form

When $n$ and $q$ are even, one has the standard and symplectic forms.

A form $g$ is *symplectic* iff $g(x, x) = 0$ for all $x$.

When $n$ is even and $q$ is odd, one has the elliptic and hyperbolic forms. The form is hyperbolic when $(-1)^{n/2} \det G$ is a square.

The standard form is hyperbolic if $(-1)^{n/2}$ is a square, and elliptic otherwise. So it is hyperbolic, unless $n \equiv 2 \pmod{4}$ and $q \equiv 3 \pmod{4}$. 
For $n = 2$ we now find for the number of nilpotent self-adjoint matrices:

- $q$ even:
  - $g$ standard: $q$
  - $g$ symplectic: $q^2$
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  - \( g \) standard: \( q \)
  - \( g \) symplectic: \( q^2 \)

Look at \( N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

Here \( G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and \( GN = (GN)^\top \) yields \( \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \), so that \( a = d \).

The trace is 0. Determinant is 0, so \( a^2 = bc \).

Now \( b \) and \( c \) can be chosen freely.
For $n = 2$ we now find for the number of nilpotent self-adjoint matrices:

$q$ even:
- $g$ standard: $q$
- $g$ symplectic: $q^2$

$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has $a = d$ and $a^2 = bc$.

(More generally, for backdiagonal $G$, the self-adjoint $N$ are those that are symmetric w.r.t. the back diagonal.)
For $n = 2$ we now find for the number of nilpotent self-adjoint matrices:

$q$ even:
- $g$ standard: $q$
- $g$ symplectic: $q^2$

$q$ odd:
- $g$ elliptic: 1
- $g$ hyperbolic: $2q - 1$

Note that $q$ is the average of 1 and $2q - 1$. 
Consider a vector space $V$ of dimension $n = 2m$ provided with a nondegenerate symplectic form $g$.

**Theorem** (Steinberg (1968), Springer (1980).) The Lie algebra $\mathfrak{sp}_{2m}$ has $q^{2m^2}$ nilpotent elements.

A matrix $A$ belongs to $Sp(2m)$ when it preserves the form, i.e., when $g(Ax, Ay) = g(x, y)$ for all $x, y$. Write $A = I + \epsilon X$, where $\epsilon^2 = 0$, to see that this means $g(x, Xy) + g(Xx, y) = 0$. For $q$ even this says that $X$ is self-adjoint.
Consider a vector space $V$ of dimension $n = 2m$ provided with a nondegenerate symplectic form $g$.

**Theorem** (Steinberg (1968), Springer (1980).) The Lie algebra $\mathfrak{sp}_{2m}$ has $q^{2m^2}$ nilpotent elements.

**Corollary** If $q$ is even, there are $q^{2m^2}$ nilpotent matrices of order $2m$ that are self-adjoint for a given nondegenerate symplectic form $g$.

This explains the $q^2$ that we got for $n = 2$. 
Consider a vector space $V$ of dimension $n = 2m$ provided with a nondegenerate symplectic form $g$.

**Theorem** (Steinberg (1968), Springer (1980).) The Lie algebra $\mathfrak{sp}_{2m}$ has $q^{2m^2}$ nilpotent elements.

**Corollary** If $q$ is even, there are $q^{2m^2}$ nilpotent matrices of order $2m$ that are self-adjoint for a given nondegenerate symplectic form $g$.

Exercise: give a direct geometric proof.
Steinberg (1968) shows for unipotent elements in algebraic groups, and Springer (1980) for nilpotent elements in the corresponding Lie algebras, that there are $q^N$ of them, where $N = |\Phi|$ is the number of roots of the root system.

The proof uses the Steinberg character and modular representation theory.
Steinberg (1968) shows for unipotent elements in algebraic groups, and Springer (1980) for nilpotent elements in the corresponding Lie algebras, that there are $q^N$ of them, where $N = |\Phi|$ is the number of roots of the root system.

For $A_{n-1}$, that is, $GL(n)$, we have $|\Phi| = n(n - 1)$, and we see again that there are $q^{n(n-1)}$ nilpotent matrices.

For $C_m$, that is, $Sp(2m)$, we have $|\Phi| = 2m^2$. If $q$ is even, there are $q^{2m^2}$ nilpotent back-symmetric matrices of order $2m$. 
$N$ is skew-symmetric when $N$ has zero diagonal and $N = -N^\top$.

For $D_m$, that is, $O^+(2m)$, we have $|\Phi| = 2m(m - 1)$. There are $q^{2m(m-1)}$ skew-symmetric nilpotent matrices of order $2m$.

For $B_n$, that is, $O(2m + 1)$, we have $|\Phi| = 2m^2$. There are $q^{2m^2}$ skew-symmetric matrices of order $2m + 1$. 
The Jordan normal form $\mathbf{N}$ of a nilpotent matrix of order $n$ has zeros on the main diagonal, and zeros and ones on the diagonal just above it. This leads to a block partition of the matrix, and to a partition of $n$.

Partitions are represented by Young diagrams $Y$.

\[
\mathbf{N} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

\[
Y = \begin{array}{ccc}
\hline
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\hline
\end{array}
\]

$e_3 \mapsto e_2 \mapsto e_1 \mapsto 0$, $e_5 \mapsto e_4 \mapsto 0$, $e_6 \mapsto 0$, $e_7 \mapsto 0$. 
Young diagrams

\[ Y = \begin{array}{cccc}
\hline
& & & \\
& & \\
& \\
\hline
\end{array} \]

\[ e_3 \mapsto e_2 \mapsto e_1 \mapsto 0, \ e_5 \mapsto e_4 \mapsto 0, \ e_6 \mapsto 0, \ e_7 \mapsto 0. \]

The map \( N \) determines a unique \( Y \). The number of rows is \( \dim \ker N \). The number of columns is the exponent of \( N \). There is a square in row \( i \) column \( j \) if \( \dim \ker N \cap \im N^{j-1} \geq i \).
Consider the Gram matrix $G = (g(u_i, u_j))_{ij}$ of ‘inner products’ of basis vectors belonging to the Young diagram $Y$, with the $u_i$ identified with the squares of the diagram. If $u_i$ has more squares to its right than $u_j$ to its left, then $g(u_i, u_j) = g(N^a u_h, u_j) = g(u_h, N^a u_j) = 0$. 

$\begin{array}{ccc}
\text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} \\
\end{array}$
If $u_i$ has more squares to its right than $u_j$ to its left, then $g(u_i, u_j) = 0$.

$$Y = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$
If $u_i$ has more squares to its right than $u_j$ to its left, then $g(u_i, u_j) = 0$.

\[
Y = \begin{array}{cccccc}
1 & 5 & 7 \\
2 & 6 \\
3 \\
4
\end{array}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & * \\
0 & . & . & . & . & . \\
0 & . & . & . & . & . \\
0 & . & . & . & . & . \\
0 & . & . & . & . & . \\
* & . & . & . & . & .
\end{bmatrix}
\]
If $u_i$ has more squares to its right than $u_j$ to its left, then $g(u_i, u_j) = 0$.

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$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & * & . & . & . & . \\ * & . & . & . & . & . \end{bmatrix}$$
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If $u_i$ has more squares to its right than $u_j$ to its left, then $g(u_i, u_j) = 0$.

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Y = \begin{bmatrix}
1 & 5 & 7 \\
2 & 6 \\
3 \\
4
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * & . \\
0 & 0 & a & b & 0 & . & . \\
0 & 0 & b & c & 0 & . & . \\
0 & 0 & 0 & 0 & * & . & . \\
0 & * & . & . & . & . & . \\
* & . & . & . & . & . & . 
\end{bmatrix}
\]
If $u_i$ has more squares to its right than $u_j$ to its left, then $g(u_i, u_j) = 0$.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & a & b & 0 & . & . \\
0 & 0 & b & c & 0 & . & . \\
0 & 0 & 0 & 0 & d & . & . \\
0 & e & . & . & . & . & . \\
\end{bmatrix}
\]

Get a transversal of nonsingular symmetric subblocks: for each group $R$ of $r$ rows of length $s$, get an $r \times r$ subblock with rows indexed by $Y_{hi}$ and columns by $Y_{h, s+1-i}$ ($h \in R$) for each $i$, $1 \leq i \leq s$. Different $i$ give the same block.
Young diagrams (2)

Another example:

\[
Y = \begin{bmatrix}
1 & 4 & 7 \\
2 & 5 \\
3 & 6
\end{bmatrix}
\]

Get a transversal of nonsingular symmetric subblocks: for each group \( R \) of \( r \) rows of length \( s \), get an \( r \times r \) subblock with rows indexed by \( Y_{hi} \) and columns by \( Y_{h,s+1-i} \) \((h \in R)\) for each \( i \), \( 1 \leq i \leq s \). Different \( i \) give the same block.
Another example:

\[
Y = \begin{bmatrix}
1 & 4 & 7 \\
2 & 5 \\
3 & 6
\end{bmatrix}
\]

Diagrams \( Y \) describe conjugacy classes of unipotent matrices. (Or, orbits of nilpotent matrices under conjugation.)
Another example:

$$Y = \begin{array}{cccccc}
1 & 4 & 7 \\
2 & 5 \\
3 & 6
\end{array}$$

If the form is symplectic, then the Gram matrix has zero diagonal. This means that each odd part of the partition has even multiplicity.
We want to count the number of self-adjoint nilpotent matrices in six cases: for odd $q$ there are the elliptic, hyperbolic, and parabolic forms, for even $q$ the symplectic and standard forms. Let us call these counts $e(2m)$, $h(2m)$, $p(2m + 1)$, $z(2m)$, $s(2m)$, $s(2m + 1)$.

**Theorem** All of $e(2m)$, $h(2m)$, $p(2m + 1)$, $z(2m)$, $s(2m)$, $s(2m + 1)$ *are polynomials in* $q$. 


So far we learned one value: $z(2m) = q^{2m^2}$.

**Theorem** $p(2m + 1) = s(2m + 1)$.

Put $a(2m) = (h(2m) + e(2m))/2$
and $d(2m) = (h(2m) - e(2m))/2$.

**Theorem** $a(2m) = s(2m)$.

In both cases, the equality is one of polynomials.
Results

**Theorem** \( p(2m + 1) = q^{2m}a(2m) + q^m d(2m) \).

**Theorem** \( p(2m + 1) = (q^{2m} - 1)a(2m) + z(2m) \).

**Theorem** \( a(2m) = q^{2m-1}p(2m - 1) \).

These settle all values recursively.
Consider $V$, with nondegenerate symmetric bilinear form $g$. The number of self-adjoint $M$ is $q^{n(n+1)/2}$.

The map $M : V \rightarrow V$ determines a unique *Fitting decomposition* $V = U \oplus W$ of $V$, where $M$ is nilpotent on $U$ and invertible on $W$.

If $u \in U$, $w \in W$, then $w = M^i w_i$ for a $w_i \in W$, and $g(u, w) = g(u, M^i w_i) = g(M^i u, w_i) = 0$ for large $i$. So $V = U \perp W$, and $U = W^\perp$, $W = U^\perp$, so that $U$ and $W$ are nondegenerate, and determine each other.
Let $N(U)$ be the number of nilpotent self-adjoint maps on $U$ (provided with the restriction of $g$ to $U$), and let $S(W)$ be the number of invertible self-adjoint maps on $W$. We proved:

$$q^{n(n+1)/2} = \sum_{U} N(U)S(U^\perp),$$

where the sum is over all nondegenerate subspaces $U$ of $V$.

By induction one finds $N(V)$. 
One finds explicit formulas for the number of nilpotent maps of given type that have a given Young diagram \( Y \) by counting pairs \((N, g)\).

E.g., for \( n = 2m + 1 \),

\[
N_s(Y) = \frac{N(Y)g_s(Y)}{g_s}
\]

\( N_s(Y) \): \# symmetric nilpotent maps of shape \( Y \)

\( N(Y) \): total \# nilpotent maps of shape \( Y \)

\( g_s \): total \# nondegenerate symmetric bilinear forms (on \( V \), where \( \dim V = n \))

\( g_s(Y) \): \# such forms for which a given \( N \) of shape \( Y \) is self-adjoint.
Counts by $Y$

Each of $N(Y)$, $g_s$, $g_s(Y)$ is easy to compute. (For $g_s(Y)$ one uses the transversal of nonsingular blocks.)

This means that all counts are known as a sum $\sum_Y N_s(Y)$ over Young diagrams. Good for checking small values. Good for proving theorems.
We have precise conjectures, but few proofs. However, there are recurrences, so all that is missing is algebraic manipulation.

The recurrences allow one to compute all counts for much larger $n$ than is possible with the sums over $Y$. 
Counts by rank

Let $p(2m + 1, r)$, $h(2m, r)$, $e(2m, r)$ count selfadjoint nilpotent matrices of rank $r$ (for odd $q$). Define $a(2m, r)$, $d(2m, r)$ as before.

**Conjectures**

(i) $p(2m + 1, 2s + 1) = (q^{2m-2s} - 1)p(2m + 1, 2s)$.

(ii) $a(2m, 2s + 1) = (q^{2m-2s-1} - 1)a(2m, 2s)$.

(iii) $d(2m, 2s) = (q^{2m-2s} - 1)d(2m, 2s - 1)$.

(iv) $(q^{2m-r} - 1)p(2m + 1, r) = (q^{2m} - 1)a(2m, r)$.

(v)

\[
p(2m + 1, 2s) = q^{s(s+1)} \prod_{i=0}^{s-1} (q^{2m-2i} - 1) \cdot \sum_{i=0}^{s} q^{(s-i)(2m-2s-1)} \binom{m-s-1+i}{i} q^2.
\]

\[
d(2m, 2s + 1) = (q-1)q^{m+s(s+1)-1} \prod_{i=1}^{s} (q^{2m-2i} - 1) \cdot \sum_{i=0}^{s} q^{(s-i)(2m-2s-3)} \binom{m-s-1+i}{i} q^2.
\]

There are similar conjectures for even $q$. 
Recursions:

**Proposition**

(i) \((q^{2m+1-r} - 1)p(2m + 1, r) = (q^{2m} - 1)p_0(2m + 1, r) + q^{2m}(q - 1)a(2m, r) + q^m(q - 1)d(2m, r)\).

(ii) \((q^{2m-r} - 1)a(2m, r) = (q^{2m-1} - 1)a_0(2m, r) + q^{m-1}(q - 1)d_0(2m, r) + q^{2m-1}(q - 1)p(2m - 1, r)\).

(iii) \((q^{2m-r} - 1)d(2m, r) = (q^{2m-1} - 1)d_0(2m, r) + q^{m-1}(q - 1)a_0(2m, r) - q^{m-1}(q - 1)p(2m - 1, r)\).

And for \(f\) any of \(p, h, e, a, d\):

(iv) \(f_0(n, r) = q^r f(n - 2, r) + (q - 1)q^{r-1} f(n - 2, r - 1) + (q^{n-r} - 1)q^{r-1} f(n - 2, r - 2)\).

Here \(f(n, r) = f_0(n, r) = 0\) for \(r < 0\) or \(r > n\) or \(r = n > 0\). As start of the recursion only \(h(0, 0) = 1\) is needed.
Let now $N_s(n, e)$ be the number of $N$ with exponent $e$.

There is information on the case with large $e$.

**Proposition** For odd $n$ we have

$$N_s(n + 2, n + 2) = q^n(q^{n+1} - 1)N_s(n, n).$$

This is $N_s(Y)$, $Y = \square\square\square\square\square\square\square$.

**Proposition** For $n$ odd, $n > 2i$, the ratio $N_s(n, n - i)/N_s(n, n)$ is independent of $n$. 
Theorem \textit{All counts are polynomials in }q.\textit{.}

Proof The sums over $Y$ are rational functions of $q$ that are integral for all $q$. $\square$

Theorem $p(2m + 1) = s(2m + 1)$.

Proof Write both counts as sums over $Y$. The parity of $q$ never plays a role. $\square$
Proofs

**Theorem** \( a(2m) = s(2m) \).

**Proof** Write as sums over \( Y \) and show termwise equality. Reduce to \( g_h(Y) - g_e(Y) = q^m g_z(Y) \).

Look at the block structure of a form \( g \).

Off-diagonal blocks contribute \( \pm \) a square to \( \det G \) and do not influence whether the form will be hyperbolic, elliptic, or symplectic. Use multiplicativity of both \( g_h - g_e \) and \( q^{n/2} g_z \) for taking orthogonal direct sums. □
**Proposition** Let $g$ be a standard symmetric bilinear form on $V$. Then

$$\#\{N \mid N \text{ self-adjoint, nilpotent}\} = \#\{(N, x) \mid N \text{ idem, } Nx = 0, \ g(x, x) \neq 0\}$$

when $n = 2m + 1$, $q$ even or odd, and when $n = 2m$, $q$ even.

**Proof** (for $n = 2m + 1$): Write as sums over $Y$, and show that the terms can be grouped so as to get equality.
The grouping is given by the map

that moves the bottom square from the rightmost odd column to form a new row of length one at the bottom. □

Proof (for $n = 2m$, $q$ even): Use the Fitting decomposition. □
Proofs

**Theorem** \( p(2m + 1) = q^{2m}a(2m) + q^{m}d(2m). \)

**Theorem** \( s(2m + 1) = (q^{2m} - 1)s(2m) + z(2m). \)

**Theorem** \( s(2m) = q^{2m-1}s(2m - 1). \)

**Proof** The first says that \( p(2m + 1) = \frac{1}{2}q^{m}(q^{m} + 1)h(2m) + \frac{1}{2}q^{m}(q^{m} - 1)e(2m). \)

All follow from the proposition above. \( \Box \)
That was all.