A construction of the Suzuki graph

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Abstract

We give an elementary construction of the rank 3 Suzuki graph, following Horiguchi, Kitazume & Nakasora.

1 Introduction

The Suzuki sporadic simple group Suz was discovered by M. Suzuki [3] as index 2 subgroup of the automorphism group of a rank 3 graph Σ, strongly regular with parameters $(v, k, \lambda, \mu) = (1782, 416, 100, 96)$, the largest graph in the Suzuki tower (of graphs for Suz:2, $G_2(4)$:2, HJ:2, $U_3(3)$:2, $L_3(2)$:2). The spectrum of Σ is 416^1, 20^{780}, $(-16)^{1001}$.

The Hoffman bound for the size of a coclique in Σ is 66, and in fact this graph has a single orbit of cocliques of this size, as was found by Kuzuta [2] and is easy to check.

If C is a 66-coclique in Σ, then each point outside C has 16 neighbours in C, so that the 1716 points outside C induce the structure of a block design with parameters 2-(66,16,96) on C. It was observed by Horiguchi, Kitazume & Nakasora [1], that this block design happens to be a 3-(66,16,21).

Something funny happens here: usually when one has a maximal coclique in a graph belonging to a nice group, then the group is transitive on the vertices of the coclique. But here the stabilizer of C in Suz:2 is $U_3(4)$:4 acting with vertex orbit sizes $1 + 65$, 2-transitively on the orbit of length 65.

In [1] the graph Σ was constructed in this setting, starting from designs on the unital in $PG(2, 16)$. We repeat their construction here, perhaps in a more elegant way.

2 A 2-(65,15,21) design and the $G_2(4)$ graph

Consider the vector space V of dimension 3 over the field $\mathbf{F}_{16}$, provided with a nondegenerate Hermitean form $\langle\cdot,\cdot\rangle$. The projective plane PV has 65 isotropic and 208 nonisotropic points. There are $208.12.1/6 = 416$ orthogonal bases (triples of orthogonal nonisotropic points).

We would like to construct the local graph Γ of Σ, a strongly regular graph with parameters $(v, k, \lambda, \mu) = (416, 100, 36, 20)$. The group $U_3(4)$:4 of semilinear transformations preserving the form acts transitively on the 416 bases, with rank 5. The suborbit sizes are 1, 15, 100, 150, 150. The graph Γ is the graph on the 416 bases obtained by taking the suborbit of size 100 for adjacency.

These suborbits can be described geometrically as follows: Given one basis $\{a, b, c\}$, the suborbit of size 15 consists of the bases that have an element in common with $\{a, b, c\}$. The first suborbit of size 150 consists of the bases that are disjoint from $\{a, b, c\}$ but contain a point orthogonal to one of a, b, c. Associated with a basis $\{a, b, c\}$ is the triangle consisting of the 15 isotropic points on the three lines ab, ac, and bc. The suborbits of sizes 1, 15, 100, 150, 150 correspond to the bases with triangles having 15, 5, 3, 2, 5 points in common, respectively. Thus, Γ can be described as the graph on the 416 triangles, adjacent when they have 3 points in common ([1]).

These 416 triangles form a 2-(65,15,21) design.

3 A 3-(66,16,21) design and the Suz graph

The set of 16 hyperbolic lines on any isotropic point y carries the structure of an affine plane of order 4. This affine plane has 20 lines of size 4, so we find 20 sets of four lines on y, and taking only the isotropic points distinct from y on these lines, we find for each y 20 sets of 16 isotropic points, and altogether 1300 such sets. Since $U_3(4)$:4 acts 2-transitively these 1300 sets will be the blocks of a 2-design with parameters 2-(65,16,75).

If we add a point ∞ to the set of isotropic points, and to the blocks (triangles) of the 2-(65,15,21) design found earlier, then the union of both designs is clearly a 2-(66,16,96), and in fact even a 3-(66,16,21) design. Indeed, the group is transitive on collinear triples of isotropic points and on noncollinear such triples so only a single equality needs to be checked.

Let $(X, \mathcal{B})$ be the 3-(66,16,21) design just constructed, with 66 points and 1716 blocks. It has block intersection numbers 0, 3, 4, 6, 16, and each block meets n_i blocks in i points, where $n_0 = 75$, $n_3 = 800$, $n_4 = 400$, $n_6 = 440$, $n_{16} = 1$.

Now construct the Suz graph with $1782 = 66 + 1716 = 1 + 65 + 416 + 1300$ vertices, where the $1 + 65$ is a 66-coclique, the 416 are the neighbours of 1 and the $416 + 1300$ (viewed as blocks) are each adjacent to their 16 points. We already saw that "having 3 points in common" was the right adjacency for the 15-point triangles, and now that ∞ has been added the construction becomes: let two blocks be adjacent when they have 4 points in common. This defines the sporadic Suzuki graph.

References

[1] N. Horiguchi, M. Kitazume & H. Nakasora, *A construction of the sporadic Suzuki graph from* $U_3(4)$, preprint, 2008.

[2] K. Kuzuta, Master Thesis, Chiba Univ., 2007 (in Japanese).

[3] M. Suzuki, *A simple group of order 448,345,497,600*, pp. 113–119 in: Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass, 1968), Benjamin, New York, 1969.