

# The graph with spectrum $14^1 2^{40} (-4)^{10} (-6)^9$

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## Abstract

We show that there is a unique graph with spectrum as in the title.

## 1 The graph $\Delta$

It was shown in [CGS] that there is a unique graph  $Z$  with spectrum  $30^1 2^{90} (-10)^{21}$  (with multiplicities written as exponents), namely the collinearity graph of the unique generalized quadrangle with parameters  $\text{GQ}(3, 9)$ . It is strongly regular with parameters  $(v, k, \lambda, \mu) = (112, 30, 2, 10)$ . Its automorphism group is  $U_4(3).D_8 \simeq \text{PGO}_6^*(3)$  (of order  $2^{10} \cdot 3^6 \cdot 5 \cdot 7$ ), where the  $*$  denotes that the form may be multiplied by a constant.

It was shown in [BH] that there is a unique graph  $Y$  with spectrum  $20^1 2^{60} (-7)^{20}$ . It is strongly regular with parameters  $(v, k, \lambda, \mu) = (81, 20, 1, 6)$ , and is the second subconstituent of  $Z$ , the subgraph induced on the set of vertices at distance 2 from a fixed vertex  $a$  of  $Z$ . Its automorphism group is  $3^4 : ((2 \times S_6).2)$  acting rank 3, the point stabilizer in  $\text{Aut}(Z)$ . One construction of  $Y$  is found by taking  $\mathbf{1}^\perp / \langle \mathbf{1} \rangle$  (where  $\mathbf{1}$  denotes the all-1 vector) inside  $\mathbf{F}_3^6$ , where two cosets are adjacent when they differ by a weight-3 vector.

Let  $\Delta$  be the second subconstituent of  $Y$ , the subgraph induced on the set of vertices at distance 2 from a fixed vertex  $b$  of  $Y$ . Then  $\Delta$  has spectrum  $14^1 2^{40} (-4)^{10} (-6)^9$  (apply Theorem 5.1 of [CGS]) and automorphism group  $(2^2 \times S_6).2$ , the stabilizer of the unordered pair  $\{a, b\}$  in  $\text{Aut}(Z)$ , twice as large as the point stabilizer of  $\text{Aut}(Y)$ . The above description of  $Y$  leads to a description of  $\Delta$  as the graph on the cosets in  $\mathbf{F}_3^6$  with coordinates (up to permutation) either  $000012 + \langle \mathbf{1} \rangle$  or  $001122 + \langle \mathbf{1} \rangle$ , where two cosets are adjacent when they differ by a weight 3 vector.

In this note we show that the graph  $\Delta$  is determined by its spectrum.

## 2 Interlacing

An important tool is the following lemma on interlacing eigenvalues ([H], Theorem 2.1 (i),(ii); see also [BCN], Theorem 3.3.1).

**Lemma 2.1** *Let  $\Gamma$  be a graph on  $n$  vertices with eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$ , and let  $\{X_1, \dots, X_m\}$  be a partition of the vertex set of  $\Gamma$  into nonempty parts. Let  $r_{ij}$  be the average number of neighbours in  $X_j$  of a vertex in  $X_i$ . Then the matrix  $R = (r_{ij})$  has real eigenvalues  $\mu_1 \geq \dots \geq \mu_m$ , which satisfy*

- (i) *(interlacing)  $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$  for  $i = 1, \dots, m$ ;*
- (ii) *if  $\mu_i = \lambda_i$ , or  $\mu_i = \lambda_{n-m+i}$  for some  $i \in \{1, \dots, m\}$ , then  $R$  has a  $\mu_i$ -eigenvector  $v = (v_1, \dots, v_m)^\top$ , such that the vector  $w \in \mathbf{R}^n$  whose entries are equal to  $v_j$  for all vertices in  $X_j$  ( $j = 1, \dots, m$ ) is a  $\mu_i$ -eigenvector of  $\Gamma$ .*

For example if  $m = 1$  it follows that the average valency  $\bar{k}$  of  $\Gamma$  is at most equal to  $\lambda_1$ , and equality implies that the all-1 vector is a  $\lambda_1$ -eigenvector of  $\Gamma$ . Since  $n\bar{k} = \sum \lambda_i^2$  it follows that  $\Gamma$  is regular of valency  $\lambda_1$  if  $n\lambda_1 = \sum \lambda_i^2$ .

### 3 Graphs cospectral to $\Delta$

Let  $\Gamma$  be a graph with the same spectrum  $14^1 \ 2^{40} \ (-4)^{10} \ (-6)^9$  as  $\Delta$ .

We shall write  $x \sim y$  ( $x \not\sim y$ ) when  $x$  is a (non)neighbour of  $y$  in  $\Gamma$ , and denote the number of common neighbours of  $x$  and  $y$  by  $\lambda(x, y)$  ( $\mu(x, y)$ ).

(i) By Lemma 2.1 we know that  $\Gamma$  is regular of valency 14. Moreover  $\Gamma$  is connected, because the multiplicity of the eigenvalue 14 equals 1.

If  $\Gamma$  has adjacency matrix  $A$ , then  $(A - 2I)(A + 4I)(A + 6I) = 72J$  so that  $(A^3)_{xx} = 8$ , and it follows that each vertex is in 4 triangles.

(ii) For a vertex  $x$ , let  $T_x$  be a set of 8 neighbours of  $x$  such that  $\{x\} \cup T_x$  contains the four triangles on  $x$ . Let  $S_x$  be the set of the remaining 6 neighbours of  $x$ , and let  $N_x$  be the set of 45 nonneighbours of  $x$ . The matrix of average row sums of  $A$ , partitioned according to  $\{\{x\}, T_x, S_x, N_x\}$  is

$$\begin{pmatrix} 0 & 8 & 6 & 0 \\ 1 & 1 & 0 & 12 \\ 1 & 0 & 0 & 13 \\ 0 & \frac{96}{45} & \frac{78}{45} & \frac{456}{45} \end{pmatrix}$$

with eigenvalues 14, 2, 0.40,  $-5.27$ . The 2-eigenspace is  $\langle (15, 3, 1, -1)^\top \rangle$ . By Lemma 2.1 it follows that the vector that is constant 15, 3, 1,  $-1$  on  $\{x\}, T_x, S_x, N_x$ , respectively, is 2-eigenvector of  $A$ . Therefore each vertex in  $T_x$  has precisely one neighbour in  $T_x$ , that is, two triangles on  $x$  have only  $x$  in common. It also follows that if  $z$  is a non-neighbour of  $x$  with  $a$  neighbours in  $T_x$  and  $b$  neighbours in  $S_x$ , then  $2a + b = 6$  while  $a + b = \mu(x, z)$ , so that  $a = 6 - \mu(x, z)$ .

(iii) The rank 10 matrix  $B = 4J - (A - 2I)(A + 6I)$  is positive semi-definite and hence can be written  $B = N^\top N$  for a  $10 \times 60$  matrix  $N$ .

Let  $\bar{x}$  be column  $x$  of  $N$ . Then  $x \mapsto \bar{x}$  is a representation of  $\Gamma$  in Euclidean 10-space, with

$$(\bar{x}, \bar{y}) = \begin{cases} 2 & \text{if } x = y \\ -\lambda(x, y) & \text{if } x \sim y \\ 4 - \mu(x, y) & \text{if } x \not\sim y \end{cases} .$$

It follows that for nonadjacent vertices  $x, y$  one has  $2 \leq \mu(x, y) \leq 6$ .

If  $\{x, y, z\}$  is a triangle, then  $\bar{x} + \bar{y} + \bar{z} = 0$  (since this sum has squared norm 0).

The matrix  $B$  satisfies  $JB = 0$  and  $AB = -4B$  and  $B^2 = 12B$  so that the rows of  $B$  are integral vectors with sum 0 and squared norm 24.

Row  $x$  of  $B$  has a 2 at the  $x$ -position, and a  $-1$  at the 8 positions  $z \in T_x$  (with  $\lambda(x, z) = 1$ ). If  $\bar{x} = \bar{y}$ , so that rows  $x$  and  $y$  of  $B$  are identical, then  $\mu(x, y) = 2$  and we see two 2's and at least fourteen  $-1$ 's in each row, and since there can be at most two more nonzero entries, the row sum is nonzero, contradiction. It follows that the representation is injective.

If  $(\bar{x}, \bar{y}) = -2$ , then  $\bar{y} = -\bar{x}$ . Given  $x$ , this happens for at most one  $y$ . It follows that a row of  $B$  has entries either  $2^1 1^8 0^{42} (-1)^8 (-2)^1$  or  $2^1 1^9 0^{39} (-1)^{11}$  (with multiplicities written as exponents).

(iv) Let us call a triangle a *line*. If  $\mu(x, y) = 3$  then the three common neighbours of  $x$  and  $y$  are joined to both  $x$  and  $y$  by a line. Now there are 24 lines not on  $x$  meeting  $T_x$ , and each  $y$  with  $\mu(x, y) = 3$  determines three such lines, so if there are 9 such points  $y$  then some line is seen twice. We find a line  $\{y, y', z\}$  with  $x \sim z$ . Now  $0 = (\bar{x}, \bar{y}) + (\bar{x}, \bar{y}') + (\bar{x}, \bar{z}) = 1 + 1 + (-1) = 1$ , contradiction. It follows that no row of  $B$  has pattern  $2^1 1^9 0^{39} (-1)^{11}$ .

(v) A set of roots (vectors of squared norm 2) with integral inner products spans a root lattice ([BCN], §3.10), so  $\Lambda = \langle \bar{x} \mid x \in V\Gamma \rangle$  is a 10-dimensional root lattice, orthogonal direct sum of summands of the form  $A_n$  ( $n \geq 1$ ),  $D_n$  ( $n \geq 4$ ),  $E_6$ ,  $E_7$ , or  $E_8$ .

(vi) The roots of the orthogonal direct sum of root lattices are the roots of the summands, so that an orthogonal direct sum decomposition of  $\Lambda$  gives a partition of  $V\Gamma$  such that  $(\bar{y}, \bar{z}) = 0$  if  $y, z$  are vertices from different parts. It follows that the three vertices of a triangle belong to the same part.

Consider the graph  $T$  with vertex set  $V\Gamma$  where two vertices  $x, y$  are adjacent when  $(\bar{x}, \bar{y}) = -1$ , i.e., when  $xy$  is an edge in a triangle of  $\Gamma$ . Given  $x$ , consider the five subsets  $S_i = \{u \in V\Gamma \mid (\bar{x}, \bar{u}) = i\}$  for  $i = 2, 1, 0, -1, -2$ . We have  $|S_2| = |S_{-2}| = 1$ ,  $|S_{-1}| = |S_1| = 8$ ,  $|S_0| = 42$ . The graph  $T$  is regular of valency 8. In  $T$ , any vertex  $y \in S_{-1}$  has 1 neighbour  $x$ , 1 neighbour in  $S_{-1}$ , 3 neighbours in  $S_1$ , and hence 3 neighbours in  $S_0$ . A vertex  $z \in S_0$  has 0 or 2  $\Gamma$ -neighbours in  $S_{-1}$ , so at most 2  $T$ -neighbours. We see that the connected component of  $T$  containing  $x$  has at least  $1 + 8 + 8 + 1 + (8 \cdot 3)/2 = 30$  vertices.

It follows that either the root lattice  $\Lambda$  is indecomposable, i.e., is  $A_{10}$  or  $D_{10}$ , or has precisely two summands. Since  $A_n$  has  $n(n+1)$  roots, and  $D_n$  has  $2n(n-1)$  roots, the possibilities in the latter case are  $A_5 + A_5$ ,  $A_5 + D_5$ ,  $D_5 + D_5$ .

(vii) Suppose  $\Lambda$  has a direct summand  $D_5$ . The root system  $D_5$  has 40 roots, and 30 occur as images of vertices in the corresponding connected component  $C$  of  $T$ . Let  $\Phi$  be the graph on the 40 roots of  $D_5$ , adjacent when they have inner product  $-1$ , and consider  $C$  a subset of the vertex set of  $\Phi$ . Let  $D$  be the set of 10 roots not in  $C$ . The graph  $\Phi$  is regular of valency 12. The valency inside  $C$  is 8, so each vertex in  $C$  has 4 neighbours in  $D$ . This gives 120 edges meeting  $D$ , so there are no internal edges in  $D$  and no two roots of  $D$  have inner product  $-1$ . Both  $\Phi$  and  $C$  are closed under  $u \mapsto -u$ , so also  $D$  is, and no two roots of  $D$  have inner product 1. Consequently,  $D$  has only inner products 2, 0,  $-2$  and consists of five mutually orthogonal pairs of opposite roots. But  $D_5$  does not contain 5 mutually orthogonal roots. Contradiction.

(viii) Consider the graph  $\Pi$  with as vertices the 30 pairs  $\pm\bar{x}$ , adjacent when they have nonzero inner product. Then  $\Pi$  has valency 8 and  $\lambda = 4$ . Using a Weetman argument (cf. [W]) we see that a connected component of  $\Pi$  has fewer than 30 vertices. It will follow that  $\Lambda \simeq A_5 + A_5$ .

As follows. For geodesics  $x_0 \sim x_1 \sim x_2 \sim \dots$  we find lower bounds  $n_i$  for the number of common neighbours of  $x_i$  and  $x_{i+1}$  at distance  $i$  from  $x_0$ . We can take  $n_1 = 2$  since two nonadjacent vertices in a 4-regular graph on 8 vertices must have at least 2 common neighbours. We can take  $n_2 = 3$  since the set of common neighbours of  $x_2$  and  $x_0$  has valency at least  $n_1 = 2$ , and hence size at least 3 (and an 8-vertex graph of degree 4 cannot have a cut set of size 2). Now the local graph at  $x_3$  has at least 4 vertices at distance 2 from  $x_0$ , and hence cannot have any at distance 4 from  $x_0$  and a connected component of  $\Pi$  has diameter at most 3 and size at most  $1 + 8 + (8 \cdot 3)/3 + (8 \cdot 2)/4 = 21$ , as desired.

(ix) Thus far, we identified the 60 vertices of  $\Gamma$  with the 60 roots of  $A_5 + A_5$ , and can recognize the triangles of  $\Gamma$ . It remains to find the edges of  $\Gamma$  that are not in a triangle.

Let  $C$  and  $D$  be the two sets of vertices belonging to the two systems  $A_5$ . Given  $x \in C$ , the 12 vertices  $y \in C$  with  $(\bar{x}, \bar{y}) = 0$  have common  $T$ -neighbours with  $x$ , so are nonadjacent to  $x$  in  $\Gamma$ . That determines the induced subgraph on  $C$  and on  $D$ , and we have to find the edges between  $C$  and  $D$ .

Suppose  $x \in C$ . If  $\bar{y} = -\bar{x}$ , then  $\mu(x, y) = 6$ , and the 6 common neighbours of  $x$  and  $y$  live in  $D$ , and form all neighbours of  $x$  in  $D$ . If  $u$  is a common neighbour of  $x$  and  $y$ , and  $\bar{v} = -\bar{u}$ , then also  $v$  is a common neighbour of  $x$  and  $y$ . This means that for the edges across we can identify pairs of opposite roots, and have a geometry with 15 points and 15 lines, where each point is on 3 lines and each line has 3 points. The points can be identified with the pairs from a 6-set. Then subgraph on the set of points is  $T(6)$ . The lines consist of three mutually disjoint pairs. This is the unique generalized quadrangle of order 2.

This proves that  $\Gamma$  is uniquely determined by its spectrum, and hence must be isomorphic to  $\Delta$ .

## References

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