Width and dual width of subsets in polynomial association schemes

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Abstract. The width of a subset C of the vertices of a distance-regular graph is the maximum distance which occurs between elements of C. Dually, the dual width of a subset in a cometric association scheme is the index of the "last" eigenspace in the Q-polynomial ordering to which the characteristic vector of C is not orthogonal. Elementary bounds are derived on these two new parameters. We show that any subset of minimal width is a completely regular code and that any subset of minimal dual width induces a cometric association scheme in the original. A variety of examples and applications are considered.

1 Introduction

Let $(X, \mathbf{A})$ denote a symmetric association scheme with d classes. Thus X is the vertex set and $\mathbf{A} = \{A_0, \ldots, A_d\}$ is the set of associate matrices, with $A_0 = I$. (For details, see the standard references [1, 3, 6].) Put $v = |X|$. Let $\{E_0, \ldots, E_d\}$, with $E_0 = \frac{1}{v}J$, denote the basis of primitive idempotents for the Bose-Mesner algebra $\mathcal{A}$ of the scheme. For $0 \le i \le d$, let v_i denote the (constant) row sum of A_i and let m_i denote the rank of E_i. Let P and Q be the first and second eigenmatrices of the scheme, so that $A_j = \sum P_{ij} E_i$, $E_j = \frac{1}{v} \sum Q_{ij} A_i$, $P_{ij}/v_j = Q_{ji}/m_i$ and $PQ = vI$.

We will make use of the following equation, due to Roos [10]:

Proposition 1 (Roos). *For any vectors $x, y \in \mathbb{R}^v$, we have*

$$\sum_{i=0}^{d} \frac{x^\top A_i y}{v v_i} A_i = \sum_{j=0}^{d} \frac{x^\top E_j y}{m_j} E_j. \tag{1}$$

Proof. $\sum_i \frac{x^\top A_i y}{v v_i} A_i = \sum_{i,j} \frac{x^\top P_{ji} E_j y}{v v_i} A_i = \sum_{i,j} \frac{x^\top Q_{ij} E_j y}{v m_j} A_i = \sum_j \frac{x^\top E_j y}{m_j} E_j.$ □

Let $C \subseteq X$ be a subset of the vertices of our association scheme and consider the case where both x and y are equal to the characteristic vector, χ, of C. Then we have

$$M_C := \sum_{i=0}^{d} \frac{\chi^\top A_i \chi}{v v_i} A_i = \sum_{j=0}^{d} \frac{\chi^\top E_j \chi}{m_j} E_j. \tag{2}$$

Since each A_i has non-negative entries, the left-hand side of (2) shows that M_C is a non-negative matrix. On the other hand, each E_j is positive semi-definite so the right-hand side of (2) shows that M_C is also positive semidefinite.

Let C be a subset of the vertices of a distance-regular graph Γ. The *width* of C is the maximum distance (in Γ) which occurs between any two elements of C. The point of departure for the present investigation is the following observation. If, in Equation (2), χ is taken to be the characteristic vector of a subset C having width w, then the matrix M_C can be expressed as a polynomial of degree exactly w in the adjacency matrix A of Γ. This polynomial has at most w roots, so we find $s^* \geq d - w$ where s^* is the dual degree (defined below). In the case of equality, we find that C must be a completely regular code in Γ.

A similar bound is obtained using (2) for a subset C of a cometric association scheme. Namely, if w^* is the largest integer j for which $E_j \chi \neq 0$, then the degree s of C is bounded below by $d - w^*$. In the case of equality, we show that C must induce a cometric association scheme inside the original.

We contrast these bounds to the well-known bounds of Delsarte on the minimum distance δ of a code in a metric scheme and on the strength t of a design in a cometric scheme, namely

$$\delta \leq 2s^* + 1, \qquad t + 1 \leq 2s + 1,$$

with the additional information that any code with $\delta \geq 2s^* - 1$ must be completely regular and any design with $t + 1 \geq 2s - 1$ must induce a cometric association scheme inside the original.

As applications related to these parameters, we are able to rule out certain antipodal covering graphs and we obtain new information about regular near polygons. We finish with a discussion of examples, including fundamental substructures in some important distance-regular graphs with classical parameters.

In fact, subsets which achieve our bounds are surprisingly common in the classical distance-regular graphs. As such sets arise within intrinsic structures such as regular semilattices, we identify these particular configurations as playing a potential role in classification theorems.

2 Width

In an association scheme $(X, \mathbf{A})$, consider a non-empty set $C \subseteq X$ with characteristic vector χ. The *degree set* of C is the set

$$\{i \neq 0 : \chi^\top A_i \chi \neq 0\}.$$

The *degree* s of C is the cardinality of this set, the number of non-identity relations which occur between members of C. The *dual degree set* of C is the set

$$\{j \neq 0 : \chi^\top E_j \chi \neq 0\}.$$

The *dual degree* s^* of C is the cardinality of this set.

The dual degree has combinatorial significance: let the *outer distribution matrix* B of C be the $|X| \times (d+1)$ matrix where the entry B_{xi} is the number of elements of C in relation i to the point x, so that $B_{xi} = e_x^\top A_i \chi$, where e_x is the characteristic vector of $\{x\}$. Then B has rank $s^* + 1$ (cf. [3], Lemma 2.5.1 (iv),(v) and Theorem 11.1.1 (i)).

Now assume that $(X, \mathbf{A})$ is an association scheme that is metric with respect to the ordering $A_0, A_1, \ldots, A_d$. Thus A_1 is the adjacency matrix of some distance-regular graph Γ and A_i is the distance-i relation for this graph. A non-empty subset C of X is called *completely regular* when $B_{xi} = B_{yi}$ whenever $d(x, C) = d(y, C)$. Let the *width* w of a non-empty subset C of X be the maximum distance between two vertices in C:

$$w = \max\{i : \chi^\top A_i \chi \neq 0\}.$$

Theorem 1. *Let $(X, \mathbf{A})$ be a metric d-class association scheme, and let C be a non-empty subset of X having width w and dual degree s^*. Then $w \geq d - s^*$. If equality holds, then C is completely regular.*

Proof. Clearly, the rank of B is not less than the number of distinct values taken by $d(x, C)$, and hence $s^* + 1 \geq d - w + 1$. (This is a special case of the result that s^* is an upper bound for the covering radius, cf. [3], Theorem 11.1.1 (ii).)

Now assume that equality holds. We have to show that B_{xi} does not depend on x but only on its distance $l = d(x, C)$ to C. Choose two points $y, z \in C$ with $d(y, z) = w$, and choose points $z_i \in X$ with $d(y, z_i) = w + d(z, z_i) = w + i$ $(0 \leq i \leq d - w)$. Then the rows of B indexed by the z_i are linearly independent and since $d - w + 1 = s^* + 1 = \operatorname{rk} B$, they span the row space of B, so that the row indexed by x is a linear combination of these. But no z_i with $i < l$ can be involved, since B_{xi} vanishes for $i < l$. And no z_i with $i > l$ can be involved, since B_{xj} vanishes for $j > l + w$ while $B_{z_i j}$ vanishes for $j > i + w$ but is nonzero for

$j = i + w$. Since the rows of B have constant row sum $|C|$ we see that the row of B indexed by x equals the row of B indexed by z_l, as desired. □

Since E_j is symmetric and idempotent we have $\chi^\top E_j \chi = ||E_j \chi||^2$ so that the vanishing of $\chi^\top E_j \chi$ is equivalent to that of $E_j \chi$. In case of equality in the theorem above we have some additional information about the j for which this happens.

Proposition 2. *Suppose* $(X, \mathbf{A})$ *is a metric association scheme where* A_1 *has eigenvalues* $\theta_0 > \theta_1 > \cdots > \theta_d$. *Suppose* C *is a non-empty subset of* X *with width* w *and dual degree* s^*, *where* $w = d - s^*$. *Let* $J = \{j : E_j \chi = 0\}$. *If* w *is even, then* J *can be partitioned into pairs of the form* $\{j, j+1\}$. *If* w *is odd, then* $d \in J$ *and* $J \setminus \{d\}$ *can be partitioned in this way.*

Proof. Consider the matrix M_C of (2). It is a linear combination of $A_0, \dots, A_w$ and hence $M_C = g(A_1)$ for some polynomial g of degree w. Since $A_1 = \sum \theta_j E_j$ and the E_j are mutually orthogonal idempotents, we find $M_C = \sum g(\theta_j) E_j$. Comparing coefficients with (2) we see that $m_j g(\theta_j) = \chi^\top E_j \chi$ so that $g(\theta_j)$ vanishes for precisely w values of j. Since g has degree w there cannot be other zeros, and since $g(\theta_i) \geq 0$ for all i (because M_C is positive semidefinite), the zeros must be distributed as described. □

3 An algebraic proof of Theorem 1

In this section, we provide a second proof of Theorem 1. The proof yields additional information concerning the case of equality in our bound.

Proof. The inequality $w \geq d - s^*$ is clear from our earlier discussion. Assume now that C is a subset of X satisfying $w = d - s^*$. Let χ denote the characteristic vector of C and order the eigenvalues of A_1 so that

$$E_1 \chi = \cdots E_w \chi = 0.$$

Let $x \in X$ with $d(x, C) = l$ and consider the vector

$$b = [B_{x,l}, \dots, B_{x,l+w}].$$

(Note that $B_{xi} = 0$ for $i < l$ and for $i > l + w$.) Since B_{xi} is the x-entry of $A_i \chi$, the x-entry of $v E_j \chi$ is $\sum_{i=0}^{d} Q_{ij} B_{xi}$. Thus b satisfies the equations

$$\sum_{i=l}^{l+w} B_{xi} = |C|, \tag{3}$$

$$\sum_{i=l}^{l+w} Q_{ij} B_{xi} = 0 \qquad (1 \leq j \leq w) \tag{4}$$

In order to prove that our code C is completely regular, it suffices to show that the above system of $w + 1$ equations is full rank.

We set the first equation aside for now and rephrase this question as a question about submatrices of the first eigenmatrix P of our metric scheme. If M is the submatrix of P obtained by restricting to rows indexed by $1, \ldots, w$, we wish to prove that any w consecutive columns of M are linearly independent. First, consider the square submatrix of M consisting of columns indexed $0, \ldots, w-1$. This can be factored as VT where V is a Vandermonde matrix and T is an invertible upper triangular matrix, so this submatrix is invertible.

For $0 \le l \le d-w$ let M_l denote the submatrix of M obtained by restricting to columns $l, l+1, \ldots, l+w$. We prove by induction on l that the last w columns of M_l are linearly independent.

Suppose that $l \ge 0$ and that the last w columns of M_l are linearly dependent. Since the first w columns of M_l are linearly independent (by induction), it follows that there is a vector y such that $y^\top M_l$ has only its first entry different from zero.

Let b be the vector defined above for a vertex x with $d(x, C) = l$. Then $M_l b = 0$ and so

$$0 = (y^\top M_l) b,$$

which implies that the first entry of b, namely B_{xl}, is zero. This contradicts our choice of x with $d(x, C) = l$.

So M_l has rank w for each $0 \le l \le d-w$. Now either the vector $[v_l, \ldots, v_{l+w}]$ is a linear combination of the rows of M_l or the system (3-4) has full rank. The former alternative implies that (3-4) is an inconsistent system; yet any vertex x at distance l from C provides a solution b. So we conclude that the system has full rank. Thus the parameters B_{xi} are uniquely determined by $d(x, C)$ and C is a completely regular code. □

We require an additional definition before presenting the next corollary. For $C \subseteq X$ with characteristic vector χ, the *dual degree set* of C is given by

$$\{j \neq 0 : E_j \chi \neq 0\}.$$

Corollary 1. *Let $(X, \mathbf{A})$ be a metric d-class association scheme, let $J \subseteq \{1, \ldots, d\}$, and let $w = |J|$. Consider submatrices N_e of the first eigenmatrix P obtained by restricting to rows indexed by elements of J and columns indexed by $e, e+1, \ldots, e+w-1$. If any N_e is singular, then X contains no set having width w and dual degree set $\{1, \ldots, d\} \setminus J$.* □

In fact, given the dual degree set of a hypothetical set C satisfying $w = d - s^*$, the above proof shows that we may solve a system of equations to obtain all of the distinct rows of the outer distribution matrix B of C. Of course the solutions must all be non-negative integers if such a set is to exist. In this way, we obtain additional feasibility conditions. Some calculations along these lines are outlined at the end of the paper.

4 Dual width

In this section we consider an association scheme $(X, \mathbf{A})$ which is cometric with respect to the ordering $E_0, \ldots, E_d$ of its primitive idempotents. For a non-empty subset C of X, the *dual width* w^* of C is defined by

$$w^* = \max\{j : \chi^\top E_j \chi \neq 0\}.$$

Theorem 2. *Let $(X, \mathbf{A})$ be a cometric d-class association scheme, and let C be a non-empty subset of X having dual width w^* and degree s. Then $w^* \geq d - s$. If equality holds, then C, together with the non-empty restrictions to C of the relations of $(X, \mathbf{A})$, is a cometric s-class association scheme.*

Proof. The inequality follows from (2) as follows. Since E_j can be expressed as a $\circ$-polynomial of degree j in E_1, the matrix M_C can be expressed as a $\circ$-polynomial of degree w^* in E_1. Such a polynomial can have at most w^* roots, so $\chi^T A_i \chi = 0$ for at most w^* values of i.

For a matrix M with rows and columns indexed by X, let $\bar{M}$ denote the submatrix of M obtained by restricting row and column indices to C. Then $\bar{\mathcal{A}} := \{\bar{M} : M \in \mathcal{A}\}$ is an $(s+1)$-dimensional vector space of symmetric matrices which contains $\bar{I}$ and $\bar{J}$ and is closed under entry-wise multiplication. Thus, to show that it is the Bose-Mesner algebra of an association scheme, it suffices to prove that $\bar{\mathcal{A}}$ is closed under ordinary matrix multiplication (see [3], Theorem 2.6.1).

Lemma 1. *For $|k - l| > w^*$ we have $\bar{E}_k \bar{E}_l = 0$.*

Proof of Lemma: Let Δ be the diagonal matrix with $\Delta_{xx} = 1$ if $x \in C$ and $\Delta_{xx} = 0$ otherwise. Using $||M||^2 = \operatorname{tr} M^\top M$ and $\operatorname{tr} AB = \operatorname{tr} BA$ and the fact that E_k and E_l are symmetric and idempotent, we find

$$||E_k \Delta E_l||^2 = \operatorname{tr} \Delta E_k \Delta E_l = \sum_{y,z \in C} (E_k \circ E_l)_{yz} = \chi^\top (E_k \circ E_l) \chi = \frac{1}{v} \sum_j q_{kl}^j \chi^\top E_j \chi.$$

For $|k - l| > w^*$ all terms on the right hand side vanish, so $E_k \Delta E_l = 0$, and $\bar{E}_k \bar{E}_l = \overline{E_k \Delta E_l} = 0$. $\square$

Let $\bar{\mathcal{E}}_j := \langle \bar{E}_0, \ldots, \bar{E}_j \rangle$. We show by induction on j that $\bar{\mathcal{A}} \bar{\mathcal{E}}_j = \bar{\mathcal{E}}_j \bar{\mathcal{A}} = \bar{\mathcal{E}}_j$ $(0 \leq j \leq s)$ and that $\bar{\mathcal{E}}_s = \bar{\mathcal{A}}$.

Step 1: *The set $\{\bar{E}_0, \ldots, \bar{E}_{j-1}, \bar{E}_{d-s+j}, \ldots, \bar{E}_d\}$ is a basis for $\bar{\mathcal{A}}$ for $0 \leq j \leq s+1$. In particular, $\bar{\mathcal{E}}_s = \bar{\mathcal{A}}$.*

Proof: First, consider $j = s+1$. Let i be the smallest index such that $\bar{E}_{i+1}$ is linearly dependent on $\bar{E}_0, \ldots, \bar{E}_i$. Then $\langle \bar{E}_0, \ldots, \bar{E}_i \rangle$ is closed under entry-wise multiplication by $\bar{E}_1$, hence contains all $\bar{E}_j$, hence equals $\bar{\mathcal{A}}$, so that $i + 1 = \dim \bar{\mathcal{A}} = s + 1$.

Now let $j = 0$. To prove that $\{\bar{E}_{d-s}, \ldots, \bar{E}_d\}$ is a basis, let i be the smallest index such that $\bar{E}_{d-i-1}$ is linearly dependent on $\bar{E}_{d-i}, \ldots, \bar{E}_d$. Then $\langle \bar{E}_{d-i}, \ldots, \bar{E}_d \rangle$ is closed under entry-wise multiplication by $\bar{E}_1$, hence contains all $\bar{E}_j$, hence equals $\bar{\mathcal{A}}$, so that $i + 1 = \dim \bar{\mathcal{A}} = s + 1$.

Now suppose $1 \leq j \leq s$. The sets $\{\bar{E}_0, \ldots, \bar{E}_{j-1}\}$ and $\{\bar{E}_{d-s+j}, \ldots, \bar{E}_d\}$ are both contained in bases, hence they are linearly independent. For the positive definite inner product $(M, N) = \operatorname{tr} M^\top N$ the two sets are orthogonal by the lemma, so their union is independent. □

Step 2:
a) *The set* $\{\bar{E}_0, \ldots, \bar{E}_{j-1}, \bar{I}, \bar{E}_{d-s+j+1}, \ldots, \bar{E}_d\}$ *is a basis for* $\bar{\mathcal{A}}$ *for* $0 \le j \le s$.
b) $\bar{\mathcal{A}}\bar{\mathcal{E}}_j = \bar{\mathcal{E}}_j\bar{\mathcal{A}} = \bar{\mathcal{E}}_j$ *for* $0 \le j \le s$.

Proof: Induct on j. For $j = -1$ part b) is true. Assume that part b) is true for $j - 1$, and write $\bar{I}$ on the basis

$$\{\bar{E}_0, \ldots, \bar{E}_{j-1}, \bar{E}_{d-s+j}, \ldots, \bar{E}_d\}.$$

If the coefficient of $\bar{E}_{d-s+j}$ in this expression is zero, then multiply both sides by $\bar{E}_j$. Using induction and the lemma, we get an expression of $\bar{E}_j$ in terms of elements of $\bar{E}_j\bar{\mathcal{E}}_{j-1} \subseteq \bar{\mathcal{E}}_{j-1}$, which is impossible by Step 1 (with $j = 0$). Thus the coefficient of $\bar{E}_{d-s+j}$ in the expression for $\bar{I}$ is nonzero, and part a) follows.

Now consider $\bar{\mathcal{A}}\bar{\mathcal{E}}_j$. Using the basis of $\bar{\mathcal{A}}$ found in part a) and induction and the lemma, we find part b). □

So far we have found that C induces an s-class association scheme. It remains to show that the scheme is cometric. Each $\bar{\mathcal{E}}_j$ is closed under multiplication and hence has a basis of mutually orthogonal idempotents. Going from $\bar{\mathcal{E}}_{j-1}$ to $\bar{\mathcal{E}}_j$ either all old minimal idempotents remain minimal, and we get one new minimal idempotent, or all old minimal idempotents except one remain minimal, while one idempotent that was minimal in $\bar{\mathcal{E}}_{j-1}$ is the sum of two minimal idempotents in $\bar{\mathcal{E}}_j$. (Proof: write the old idempotents on the basis of the new minimal idempotents. Then all coefficients are 0 or 1, and mutually orthogonal idempotents have the 1's in different places.) In our case we can rule out the latter possibility: If $F = F' + F''$, where F is a minimal idempotent in $\bar{\mathcal{E}}_{j-1}$ and F', F'' are minimal idempotents in $\bar{\mathcal{E}}_j$, then multiply by $\bar{E}_{d-s+j}$ to find $\bar{E}_{d-s+j}F' = -\bar{E}_{d-s+j}F''$, so that

$$\bar{E}_{d-s+j}F' = \bar{E}_{d-s+j}F'F' = -\bar{E}_{d-s+j}F''F' = 0,$$

and hence $\bar{E}_{d-s+j}\bar{\mathcal{E}}_j = 0$, impossible. Thus, the minimal idempotents of $\bar{\mathcal{E}}_{j-1}$ remain minimal in $\bar{\mathcal{E}}_j$, and we find that $\bar{\mathcal{A}}$ has minimal idempotents $F_0, \ldots, F_s$ with $\bar{\mathcal{E}}_j = \langle F_0, \ldots, F_j \rangle$ for each j. Since the elements of $\bar{\mathcal{E}}_j \setminus \bar{\mathcal{E}}_{j-1}$ are $\circ$-polynomials of degree j in $\bar{E}_1$ (and hence in F_1), the scheme on C is cometric. □

Proposition 3. *Suppose* $(X, \mathbf{A})$ *is a cometric association scheme with respect to* E_1 *where*

$$E_1 = \sum_{h=0}^{d} \sigma_h A_h$$

and $\sigma_0 > \sigma_1 > \cdots > \sigma_d$. *Suppose* C *is a non-empty subset of* X *with dual width* w *and degree* s, *where* $w^* = d - s$. *Let* $H = \{h : \chi^\top A_h \chi = 0\}$. *If* w^* *is even, then* H *can be partitioned into pairs of the form* $\{h, h+1\}$. *If* w^* *is odd, then* $d \in H$ *and* $H \setminus \{d\}$ *can be partitioned in this way.*

Proof. Analogous to the proof of Proposition 2. □

5 The case $w + w^* = d$

In this section, we consider some subsets which achieve equality in both the bound of Theorem 1 and the bound of Theorem 2. Throughout this section, $(X, \mathbf{A})$ denotes a metric and cometric association scheme. For a subset C of X, we have the following inequalities:

$$s \leq w, \qquad s^* \leq w^*,$$

$$w + s^* \geq d, \qquad w^* + s \geq d.$$

It follows that $w + w^* \geq d$ and if $w + w^* = d$, then equality is achieved in each of the four inequalities above as well. (Note that the perfect code in the Hamming graph $H(4, 3)$ satisfies $w + s^* = d = w^* + s$, yet $w = w^* = 3$.)

Theorem 3. *If $C \subseteq X$ satisfies $w + w^* = d$, then C induces a cometric association scheme in $(X, \mathbf{A})$. If this induced scheme is primitive, it is metric as well.*

Proof. The first statement follows immediately from Theorem 2. Let Δ be the subgraph of Γ induced by C. Since the induced scheme is primitive, Δ is connected. Since $s = w$, we see that distance in Δ is the same as distance in Γ. In particular, $\bar{A}_i$, $0 \leq i \leq w$, is the matrix of the distance-i relation for Δ. Therefore, Δ is distance-regular. □

Suppose $(X, \mathbf{A})$ is a metric/cometric translation scheme [3, p65] corresponding to a distance-regular graph Γ (e.g. a Hamming graph). Suppose further that C is a subgroup of the abelian group X satisfying $w + w^* = d$. Then, as in [3, Corollary 11.1.7], the coset graph Γ/C is distance-regular.

Theorem 4. *If C is an abelian subgroup of a cometric translation distance-regular graph Γ satisfying $w + w^* = d$ and the quotient graph Γ/C is primitive, then it is also cometric.*

Proof. Observe that the dual degree set of C is $\{1, \ldots, w^*\}$. Each coset of C represents a single vertex of the quotient scheme. Choose a set of coset representatives $\{x_h\}$ and let M be the 01-matrix with rows indexed by cosets and columns indexed by vertices, having a one in position (C', x_h) if x_h is the chosen representative for coset C'. For $0 \leq j \leq w^*$, the column of the j^{th} primitive idempotent of the quotient scheme indexed by C is then $u_j = \alpha_j M E_j \chi$ for some nonzero scalar α_j. We have

$$u_i \circ u_j = \frac{|C|}{|X|} \sum_{k=0}^{w^*} \tilde{q}_{i,j}^k \, u_k$$

where $\tilde{q}_{i,j}^k$ is the Krein parameter of the quotient scheme.

As C is completely regular, the space

$$\langle E_0\chi, E_1\chi, \ldots, E_{w^*}\chi \rangle$$

is closed under entry-wise multiplication. Since the original association scheme is cometric, we have

$$(E_i\chi) \circ (E_j\chi) \in \langle E_0\chi, E_1\chi, \ldots, E_{i+j}\chi\rangle \quad (5)$$

using Proposition II.8.3(i) in [1]. This means that $\tilde{q}_{i,j}^k$ will vanish if $k > i + j$. It remains only to show that $\tilde{q}_{i,j}^{i+j} > 0$ whenever $i + j \le w^*$. But this is guaranteed by primitivity. □

Some of the most important families of association schemes are associated to regular semilattices (for a definition, see [5]). In such schemes, each object in the semilattice gives rise to a code in the association scheme which achieves equality in all of our bounds. For simplicity, we assume that the semilattice has exactly d levels; i.e., any object has rank r for some $0 \le r \le d$. The definition of a regular semilattice is rather cumbersome and the proof of the following theorem involves little more than this definition, so we omit both.

Theorem 5. *Let $(\mathcal{L}, \preceq)$ be a regular semilattice [5] with its induced metric/cometric association scheme on the set X of elements of maximal rank. Let $t \in \mathcal{L}$ be an object of rank $w^* \le d$. Then the set*

$$C = \{x \in X : t \preceq x\}$$

has dual width w^* and width $w = d - w^*$. □

We will say that a code C arising in this way is of *semilattice type*. This theorem gives us examples in the Hamming and Johnson graphs as well as their respective q-analogues, the bilinear forms graphs and Grassman graphs. In each case, the set C induces a metric/cometric scheme belonging to the same family as the original (with all classical parameters preserved except the diameter).

For the Hamming and Johnson graphs, we may use results of Meyerowitz to obtain a complete classification of subsets C satisfying $w + w^* = d$.

Theorem 6 (Meyerowitz [9]). *Let C be a completely regular code of strength zero in the Johnson graph $J(v, d)$ defined on point set $\Omega = \{1, \ldots, v\}$. Then there exists a subset $T \subseteq \Omega$ such that either $C = \{x : x \subseteq T\}$ or $C = \{x : T \subseteq x\}$.*

Proof. Suppose that $J(v, d)$ is defined on the collection X of d-element subsets of the v-set Ω. Let W be the 01-matrix with columns indexed by Ω, rows indexed by X, and having a one in position (x, a) if $a \in x$. Then the column space of W is $V_0 \oplus V_1$ ([4, p47]) and hence $(E_0 + E_1)\chi = W\xi$ for some vector ξ with entries indexed by Ω. For $x \in X$, we know that the x entry of $E_1\chi$ depends only on $h = d(x, C)$: denote this value by ω_h. Moreover, from [7], we have

$$\omega_0 > \omega_1 > \omega_2 > \cdots .$$

On the other hand,

$$\omega_h + \frac{|C|}{v} = \sum_{a \in x} \xi_a.$$

Now if we consider adjacent vertices $x \in C_i$ and $y \in C_j$, say $y = x \cup \{a\} - \{b\}$, then

$$\omega_i - \omega_j = (E_1\chi)_x - (E_1\chi)_y = \xi_a - \xi_b.$$

So if $x \in C_0$, there are only two possible values for $\xi_a - \xi_b$ where $a \in x$, $b \notin x$. Thus ξ is constant on either x or $\Omega - x$. It quickly follows that ξ takes on exactly two values on Ω, thereby partitioning it into two sets, T and $\Omega - T$. Then C_0 is clearly the set of vertices x having maximal incidence with T. □

Theorem 7 (Meyerowitz, unpublished). *Let C be a completely regular code of strength zero in the Hamming graph $H(d,q)$ defined over the alphabet $\mathcal{Q}$ Then there exist coordinate indices $1 \le i_1 < \cdots < i_{d-w} \le d$ and elements $a_1, \ldots, a_{d-w} \in \mathcal{Q}$ such that $C = \{x : x_{i_1} = a_1, \ldots, x_{i_{d-w}} = a_{d-w}\}$.*

Sketch of Proof: The vertex set X is given as $\mathcal{Q}^n$ where $|\mathcal{Q}| = q$. Let W denote the 01-matrix with rows indexed by X and columns indexed by $\{1, \ldots, n\} \times \mathcal{Q}$ wherein the entry in row x column (i,a) is one if $x_i = a$. Then it has been shown [5] that the column space of W is $V_0 \oplus V_1$ and each vector in V_1 can be uniquely expressed in the form $W\xi$ where $\sum_a \xi_{i,a} = 0$ for each i, $1 \le i \le n$. Now write $E_1\chi = W\xi$. As in the previous proof, there exist constants $\omega_0 > \omega_1 > \cdots$ such that, for $x \in C_h$, the x entry of $E_1\chi$ is equal to ω_h. Now consider $x \in C_0$ and y a neighbor of x in $H(d,q)$. Then $(E_1\chi)_x - (E_1\chi)_y = \xi_{i,a} - \xi_{i,b}$ for some coordinate i and some $a \neq b$ in $\mathcal{Q}$. Since this difference can take on at most two values, we find that, for fixed i, $\xi_{i,a}$ takes at most two values. Testing a variety of adjacent pairs x and y, we eventually find that, for a coordinate i, either ξ is constant on pairs (i,a) or there is one exceptional value $a \in \mathcal{Q}$ with $\xi_{i,a} > \xi_{i,b}$ for $b \neq a$ and the remaining values $\xi_{i,b}$ all coincide. □

Theorem 8. *1. If $(X, \mathcal{A})$ is a Hamming scheme $H(d,q)$ and C is a subset of $X = \mathcal{Q}^d$ with $w + w^* = d$, then C is isomorphic to a set of the form*

$$C' = \{x \in X : x_i = 0 \text{ for } \ i \le d - w\}$$

where $0 \in \mathcal{Q}$;

2. If $(X, \mathcal{A})$ is a Johnson scheme $J(v,d)$ defined on point set $\Omega = \{1, \ldots, v\}$ and C is a subset of X with $w + w^* = d$, then C is isomorphic to one of the two sets

$$C_1 = \{x \in X : \{1, \ldots, d-w\} \subseteq x\} \qquad (v \ge 2d)$$
$$C_2 = \{x \in X : x \subseteq \{1, \ldots, d+w\}\} \qquad (v = 2d \text{ only}).$$

Proof. If C satisfies $w + w^* = d$, then C must be a completely regular code by Theorem 1. But $s^* = w^*$ implies that $E_1\chi \neq 0$. So in these families with natural Q-polynomial ordering, C is a completely regular code of strength zero. Applying Theorems 7 and 6, we obtain our result. □

6 Antipodal covers

Delsarte showed that a clique C in a distance-regular graph Γ has size at most $1 - k/\theta_d$ where k is the valency and θ_d is the smallest eigenvalue of Γ. (See [3, Proposition 1.3.2] or Equation (3.23) in [4].) A clique attaining this bound is called a *Delsarte clique*. It is easy to verify that a Delsarte clique has dual degree $d-1$ and a clique which is not a Delsarte clique has dual degree d. Delsarte cliques are rather common. For example, all Hamming, Johnson, Grassmann, bilinear forms and dual polar graphs contain such cliques. Corollary 2 below implies that graphs with a Delsarte clique cannot have an antipodal cover of odd diameter.

Theorem 9. *Suppose Γ is a distance-regular graph of diameter d containing a code C having width w and dual degree s^*. If Γ has an antipodal cover of diameter D, where $D > 3w$, then $w \geq D - 1 - 2s^*$.*

Proof. Let Δ be an antipodal r-cover of Γ of diameter D. (Then $D \in \{2d, 2d+1\}$.) The condition $3w < D$ ensures that we can lift C to a code C' in Δ that is isometric to C: pick $x \in C$ and fix x' arbitrarily in the fibre of x; for each $y \in C$ let y' be the point in the fibre of y at distance at most w from x' (unique since $2w < D$); now if $y, z \in C$ and $d(y, z) = i$, then $d(y', z') \in \{i, D - i\}$ and since $D - w > 2w$ the triangle inequality implies $d(y', z') = i$.

Let Γ and Δ have distance-j matrices A_j and B_j, respectively, and let χ, χ' be the characteristic vectors of C and C'.

Consider the matrix M_C of (2) and its analog $M_{C'}$. Since $w < D/2$, the polynomial expressing B_j in terms of B_1 is the same as the polynomial expressing A_j in terms of A_1 for $0 \leq j \leq w$. From the previous paragraph, we have $\chi'^\top B_j \chi' = \chi^\top A_j \chi$. So we find that if g is the polynomial of degree w for which $M_C = g(A_1)$, then $M_{C'} = \frac{1}{r} g(B_1)$ using the fact that Δ has r times as many vertices as Γ.

Let $\theta_0 > \theta_1 > \cdots > \theta_d$ be the distinct eigenvalues of Γ (that is, of A_1), and let $\tau_0 > \tau_1 > \cdots > \tau_D$ be those of Δ (that is, of B_1). Then $\theta_j = \tau_{2j}$ $(0 \leq j \leq d)$ (see [3], p. 142). Since $M_{C'}$ is positive semi-definite, with eigenvalues $g(\tau_i)$ $(0 \leq i \leq D)$ not more than about half of the zeros of g can be among the θ_j: we find for each zero θ_j of g one more in $[\tau_{2j-1}, \tau_{2j+1}]$ (counting multiplicities of roots) except possibly in case $j = d$ and $D = 2d$. Thus, if $D = 2d + 1$ then $w \geq 2(d - s^*)$, and if $D = 2d$ then $w \geq 2(d - s^*) - 1$. □

Note: The hypothesis $D > 3w$ can be weakened to $D > 2w + a$ where $\cup_{1 \leq j \leq a} \Gamma_j$ induces a connected graph on C. (Here, Γ_j is the distance-j graph of Γ.)

Corollary 2. *Under the hypotheses of the theorem, if $w = d - s^*$ and $w > 0$ then $w = 1$ and $D = 2d$.* □

It is not true that the existence of any subset satisfying $w = d - s^*$ precludes the existence of a cover; the following example illustrates the need for additional conditions of the sort appearing in the above theorem.

Example 1. Let Γ be the folded 6-cube of diameter $d = 3$. It has an antipodal cover of diameter $D = 6$. Γ is bipartite. Let C be one of the bipartite classes. Then C has width $w = 2$ and dual degree $s^* = 1$.

7 Regular near polygons

A *regular near polygon* is a distance-regular graph Γ, of diameter d say, which contains no induced subgraph isomorphic to $K_4 - e$ (i.e., a graph on four vertices with all pairs adjacent but one) and in which any maximal clique C contains a unique nearest vertex to any vertex x in Γ. (Compare [3, Sec. 6.4].) A regular near polygon is *thick* if its singular lines have size at least three. If Γ is thick and $c_2 \geq 2$, then a result of Yanushka [11] guarantees that any two vertices at distance two in Γ lie in a unique common *quad* (geodetically closed subgraph of diameter two — which is necessarily the collinearity graph of a generalized quadrangle). We now use the existence of quads to obtain spectral information about Γ.

Theorem 10. *Let Γ be a thick regular near $2d$-gon with quads. Then the second smallest eigenvalue of Γ satisfies*

$$\theta_{d-1} \geq a_1 + 1 - \frac{b_1}{(a_1+1)(c_2-1)}.$$

Equality holds in this bound if and only if every quad in Γ achieves the bound of Theorem 1 with equality.

Proof. Let C be the vertex set of a quad in Γ. The inner distribution of C is

$$\mathbf{a} = [1, \quad c_2(a_1+1), \quad (c_2-1)(a_1+1)^2, \quad 0, \ldots, 0].$$

The condition $\chi^\top E_j \chi \geq 0$ gives

$$1 + c_2(a_1+1)\omega_1 + (c_2-1)(a_1+1)^2\omega_2 \geq 0$$

where

$$\omega_1 = \frac{\theta_j}{k}, \qquad \omega_2 = \frac{\theta_j^2 - a_1\theta_j - k}{kb_1}.$$

So there can be no eigenvalue in the open interval

$$\left(\frac{-k}{a_1+1}, a_1 + 1 - \frac{b_1}{(a_1+1)(c_2-1)} \right).$$

Now set $k/(a_1+1) = t+1$, the number of lines through any point in the geometry. It is well-known that the smallest eigenvalue of Γ is $-(t+1)$. So the second smallest eigenvalue must satisfy

$$\theta_{d-1} \geq a_1 + 1 - \frac{b_1}{(a_1+1)(c_2-1)}.$$

It is now easy to check that a quad C in Γ satisfies $\chi^\top E_d \chi = 0$. So, by Proposition 2, C meets the bound of Theorem 1 if and only if $\chi^\top E_{d-1}\chi = 0$, i.e., precisely when our bound on θ_{d-1} is tight. If this holds for one quad in Γ, then it holds for all quads. □

One example where this is the case is the binary Hamming scheme $H(d,2)$ ($\theta_{d-1} = 2 - d$). More interesting perhaps are the quads in the dual polar spaces.

Example 2. If Γ is (the point graph of) a dual polar space, any two vertices x and y at distance two in Γ determine a unique quad, which is also a dual polar space. A simple calculation shows that all such quads meet the bound of Theorem 1 and, as Γ is cometric with respect to the standard ordering of its eigenvalues, we have $w + w^* = d$. So the bound $w^* \geq d - s$ is tight as well.

8 Examples

Let us now discuss further examples where equality holds in the bounds we have given. Since we anticipate that such sets may play a role in the further study of distance-regular graphs, we tend to be rather inclusive here.

If $(X, \mathcal{A})$ is a metric association scheme and $C \subseteq X$ satisfies $w = d - s^*$, then we say that C is *w-narrow* or simply *narrow*. Similarly, if $(X, \mathcal{A})$ is cometric and C satisfies $w^* = d - s$, then we say that C is *w^*-dual narrow* or *dual narrow*.

We have already seen examples of narrow and dual narrow subsets in Hamming and Johnson graphs as well as their q-analogues.

Example 3. If Γ is a distance-regular graph with valency k and smallest eigenvalue θ_d, then any clique C having cardinality $1 - k/\theta_d$ is 1-narrow (Delsarte clique). For example, any edge in a bipartite distance-regular graph has this property. Moreover, if Γ is cometric with respect to the natural ordering of its eigenvalues, then C is $(d-1)$-dual narrow as well.

Example 4. Let Γ be a bipartite distance-regular graph of odd diameter and let $C \subseteq X$ be one colour class of vertices of Γ. Then C is $(d-1)$-narrow.

Example 5. Let $C_3 = \{000, 011, 101, 110\}$ be the set of even weight words in $H(3, 2)$. Then, for any $k \geq 1$ and any $\ell \geq 0$, $C = C_3^k \times \{0, 1\}^\ell$ is a completely regular code in $H(3k + \ell, 2)$ having width $w = 2k + \ell$ and dual degree $s^* = k$. The dual width of such a code is $3k$ and the degree is $k + \ell$, so these examples meet only one of our bounds.

Example 6. Let $C_3^\perp = \{000, 111\}$. Then, for $k \geq 1$ and $\ell \geq 0$, $(C_3^\perp)^k \times \{0\}^\ell$ is a code in $H(3k + \ell, 2)$ having dual width $w^* = 2k + \ell$ and degree $s = k$. Clearly the association scheme induced by C is isomorphic to $H(k, 2)$.

Example 7. In the d-cube $H(d, 2)$, let x be any vertex having odd Hamming weight, then $C = \{0, x\}$ has degree one and dual width $d - 1$. The dual code $C^\perp$ satisfies $w = d - s^*$ with $w = d - 1$.

Example 8. In the halved 6-cube, one can find both the 4×4 grid and Shrikhande's graph as subgraphs. In both cases, the vertex set C is 2-narrow with dual degree set $\{3\}$.

Example 9. In the halved 7-cube, the simplex code (dual to the binary Hamming code) is 2-narrow with dual degree set $\{3\}$.

Example 10. The line graph of Petersen's graph is an antipodal cover of the complete graph K_5 with antipodal fibres of size three. It is possible (in a unique way) to choose one vertex from each antipodal class to obtain a code C having any two vertices at distance two. This set has $w = 2$ and $s^* = 1$.

Example 11. It is known that a *linked system of symmetric designs* [8] gives rise to an imprimitive cometric 3-class association scheme. The vertex set X is naturally partitioned into m sets of objects

$$X = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_m$$

and in the graph Γ, given by A_1 say, we have the incidence graph of a square 2-design between any two of these. The graph corresponding to A_3 is complete on each $\mathcal{P}_i$ and has no other edges. Under any Q-polynomial ordering, the mapping from X into the eigenspace V_3 is non-injective with the vertices in $\mathcal{P}_i$ all mapping to the same point and the m images so obtained forming a regular simplex in $\mathbb{R}^{m-1}$.

Suppose $C \subseteq X$ has exactly one element from each $\mathcal{P}_i$ and that any two elements of C are adjacent in Γ. Then C has $w^* = 2$ and $s = 1$.

Example 12. A cometric association scheme $(X, \mathbf{A})$ is *Q-bipartite* [3, p241] if its Krein parameters satisfy $q_{ij}^k = 0$ whenever $i + j + k$ is odd. In such a scheme, each fibre in the Q-bipartite system of imprimitivity has size two and there are $|X|/2$ such fibres. (Proof: E_1 has no repeated columns, but E_2 does. Since $E_2 = \frac{1}{q_{11}^2}(E_1 \circ E_1 - m_1 E_0)$, the angle between two vectors $E_1 e_x$ and $E_1 e_y$ with x and y in the same fibre is either zero or π.) Any fibre in a Q-bipartite d-class scheme (d odd) has $w^* = d - 1$ and $s = 1$. As a special case, if Γ is a cometric antipodal distance-regular graph of odd diameter, then any antipodal class in Γ is $(d-1)$-dual narrow.

Examples having small width in the Hamming schemes For small width, most w-narrow codes in the Hamming graphs are of semilattice type. Suppose C is a subset of the vertex set of $H(d, q)$ and satisfies $w + s^* = d$. If $E_1\chi \neq 0$, then C is a completely regular code of strength zero and, applying Theorem 7, C is of semilattice type and satisfies $w + w^* = d$.

A 1-narrow code in $H(d, q)$ is a Delsarte clique and these are of semilattice type. Next assume that C is a 2-narrow code in $H(d, q)$ which is not of semilattice type. Then $E_1\chi = 0$. If we write the inner distribution of C as $\mathbf{a} = [1, \phi_1, \phi_2, 0, \ldots, 0]$ and use Proposition 2, from which $E_1\chi = 0$ forces $E_2\chi = 0$, we obtain two equations in the unknowns ϕ_1 and ϕ_2 and we find that ϕ_1 is negative for all values $d \geq 2$, $q \geq 2$ with one exception: $d = 3$, $q = 2$. Clearly the words of even weight in $H(3, 2)$ give the unique code with these properties.

By the same token, if C is a 3-narrow subset in $H(d, q)$, then either C has strength zero or we have $E_1\chi = E_2\chi = E_d\chi = 0$. One example is the perfect Hamming code in $H(4, 3)$. (Note that all Hamming codes have $s^* = 1$, but this is the only Hamming code with $w < d$.) Another example is found in $H(4, 2)$, namely

$$C = \{000, 011, 101, 110\} \times \{0, 1\}.$$

If C is any other example with inner distribution $\mathbf{a} = [1, \phi_1, \phi_2, \phi_3, 0, \ldots, 0]$, then we may solve for the ϕ_i and, knowing $\phi_2 \geq 0$, we obtain

$$(d, q) \in \{(4, 2), (4, 3), (5, 2), (6, 2)\}.$$

Our two examples arise from the first two parameter sets while for the last two parameter sets, we find $\phi_1 < 0$.

Examples having small width in the Johnson schemes Similar considerations allow us to classify 2-narrow sets in the Johnson graphs $J(v, k)$. Aside from the examples of semilattice type, we find two designs, namely a 2-design with ten blocks in $J(6, 3)$ and the Fano plane in $J(7, 3)$.

There is only one example of a 3-narrow set in $J(v, k)$ which is not of semilattice type. This can be constructed as follows. Consider a copy of the affine plane $AG(2, 3)$. Impose a cyclic ordering on the three lines in each of the four parallel classes. As blocks of our design C, take all antiflags $\{P\} \cup \ell$ in $AG(2, 3)$ having the property that the lone point P lies on the line following ℓ in the cyclic ordering. With a bit of work, one may show that this is the only design having width three and dual degree one in $J(9, 4)$.

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