

Two distance-regular graphs

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Abstract

We construct two families of distance-regular graphs, namely the subgraph of the dual polar graph of type $B_3(q)$ induced on the vertices far from a fixed point, and the subgraph of the dual polar graph of type $D_4(q)$ induced on the vertices far from a fixed edge. The latter is the extended bipartite double of the former.

1 The extended bipartite double

We shall use \sim to indicate adjacency in a graph. For notation and definitions of concepts related to distance-regular graphs, see [BCN]. We repeat the definition of extended bipartite double.

The *bipartite double* of a graph Γ with vertex set X is the graph with vertex set $\{x^+, x^- \mid x \in X\}$ and adjacencies $x^\delta \sim y^\epsilon$ iff $\delta\epsilon = -1$ and $x \sim y$. The bipartite double of a graph Γ is bipartite, and it is connected iff Γ is connected and not bipartite. If Γ has spectrum Φ , then its bipartite double has spectrum $(-\Phi) \cup \Phi$. See also [BCN], Theorem 1.11.1.

The *extended bipartite double* of a graph Γ with vertex set X is the graph with vertex set $\{x^+, x^- \mid x \in X\}$, and the same adjacencies as the bipartite double, except that also $x^- \sim x^+$ for all $x \in X$. The extended bipartite double of a graph Γ is bipartite, and it is connected iff Γ is connected. If Γ has spectrum Φ , then its extended bipartite double has spectrum $(-\Phi - 1) \cup (\Phi + 1)$. See also [BCN], Theorem 1.11.2.

2 Far from an edge in the dual polar graph of type $D_4(q)$

Let V be a vector space of dimension 8 over a field F , provided with a nondegenerate quadratic form of maximal Witt index. The maximal totally isotropic subspaces of V (of dimension 4) fall into two families \mathcal{F}_1 and \mathcal{F}_2 , where the dimension of the intersection of two elements of the same family is even (4 or 2 or 0) and the dimension of the intersection of two elements of different families is odd (3 or 1).

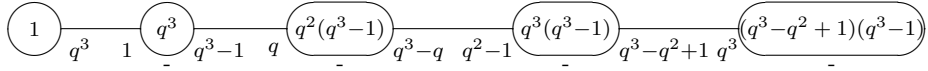
The geometry of the totally isotropic subspaces of V , where $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ are incident when $\dim A \cap B = 3$ and otherwise incidence is symmetrized inclusion, is known as the geometry $D_4(F)$. The bipartite incidence graph on

the maximal totally isotropic subspaces is known as the dual polar graph of type $D_4(F)$.

Below we take $F = \mathbf{F}_q$, the finite field with q elements, so that graph and geometry are finite. We shall use projective terminology, so that 1-spaces, 2-spaces and 3-spaces are called points, lines and planes. Two subspaces are called disjoint when they have no point in common, i.e., when the intersection has dimension 0.

Proposition 2.1 *Let Γ be the dual polar graph of type $D_4(\mathbf{F}_q)$. Fix elements $A_0 \in \mathcal{F}_1$ and $B_0 \in \mathcal{F}_2$ with $A_0 \sim B_0$. Let Δ be the subgraph of Γ induced on the set of vertices disjoint from A_0 or B_0 . Then Δ is distance-regular with intersection array $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1, q^3\}$.*

The distance distribution diagram is



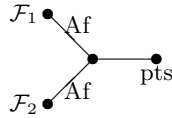
Proof: There are q^6 elements $A \in \mathcal{F}_1$ disjoint from A_0 and the same number of $B \in \mathcal{F}_2$ disjoint from B_0 , so that Δ has $2q^6$ vertices.

Given $A \in \mathcal{F}_1$, there are $q^3 + q^2 + q + 1$ elements $B \in \mathcal{F}_2$ incident to it. Of these, $q^2 + q + 1$ contain the point $A \cap B_0$ and hence are not vertices of Δ . So, Δ has valency q^3 .

Two vertices $A, A' \in \mathcal{F}_1$ have distance 2 in Δ if and only if they meet in a line, and the line $L = A \cap A'$ is disjoint from B_0 . If this is the case, then L is in $q + 1$ elements $B \in \mathcal{F}_2$, one of which meets B_0 , so that A and A' have $c_2 = q$ common neighbours in Δ .

Given vertices $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ that are nonadjacent, i.e., that meet in a single point P , the neighbours A' of B at distance 2 to A in Δ correspond to the lines L on P in A disjoint from B_0 and nonorthogonal to the point $A_0 \cap B$. There are $q^2 + q + 1$ lines L on P in A , $q + 1$ of which are orthogonal to the point $A_0 \cap B$, and one further of which meets B_0 . (Note that the points $A_0 \cap B$ and $A \cap B_0$ are nonorthogonal since neither point is in the plane $A_0 \cap B_0$ and V does not contain totally isotropic 5-spaces.) It follows that $c_3 = q^2 - 1$, and also that Δ has diameter 4, and is distance-regular. \square

The geometry induced by the incidence relation of $D_4(F)$ on the vertices of Δ , together with the points and lines contained in the planes disjoint from $A_0 \cup B_0$, has Buekenhout-Tits diagram (cf. [P])



that is, the residue of an object $A \in \mathcal{F}_1$ is an affine 3-space, where the objects incident to A in \mathcal{F}_2 play the rôle of points. Similar things hold more generally for $D_n(F)$ with arbitrary n , and even more generally for all diagrams of spherical type. See also [BB], Theorem 6.1.

Let P be a nonsingular point, and let ϕ be the reflection in the hyperplane $H = P^\perp$. Then ϕ is an element of order two of the orthogonal group that fixes H pointwise, and consequently interchanges \mathcal{F}_1 and \mathcal{F}_2 . For each $A \in \mathcal{F}_1$ we

have $\phi(A) \sim A$. The quotient Γ/ϕ is the dual polar graph of type $B_3(q)$, and we see that more generally the dual polar graph of type $D_{m+1}(q)$ is the extended bipartite double of the dual polar graph of type $B_m(q)$. The quotient Δ/ϕ is a new distance-regular graph discussed in the next section. It is the subgraph consisting of the vertices at maximal distance from a given point in the dual polar graph of type $B_3(q)$. For even q we have $B_3(q) = C_3(q)$, and it follows that the symmetric bilinear forms graph on \mathbf{F}_q^3 is distance-regular, see [BCN] Proposition 9.5.10 and the diagram there on p. 286.

3 Far from a point in the dual polar graph of type $B_3(q)$

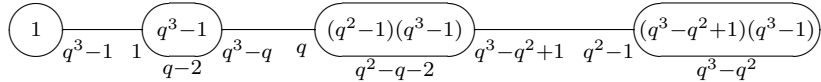
First a very explicit version of the graph of this section.

Proposition 3.1 (i) *Let W be a vector space of dimension 3 over the field \mathbf{F}_q , provided with an outer product \times . Let Z be the graph with vertex set $W \times W$ where $(u, u') \sim (v, v')$ if and only if $(u, u') \neq (v, v')$ and $u \times v + u' - v' = 0$. Then Z is distance-regular of diameter 3 on q^6 vertices. It has intersection array $\{q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1\}$ and eigenvalues $q^3 - 1, q^2 - 1, -1, -q^2 - 1$ with multiplicities $1, \frac{1}{2}q(q+1)(q^3 - 1), (q^3 - q^2 + 1)(q^3 - 1), \frac{1}{2}q(q-1)(q^3 - 1)$, respectively.*

(ii) *The extended bipartite double \hat{Z} of Z is distance-regular with intersection array $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1, q^3\}$ and eigenvalues $\pm q^3, \pm q^2, 0$ with multiplicities $1, q^2(q^3 - 1), 2(q^3 - q^2 + 1)(q^3 - 1)$, respectively.*

(iii) *The distance-1-or-2 graph $Z_1 \cup Z_2$ of Z , which is the halved graph of \hat{Z} , is strongly regular with parameters $(v, k, \lambda, \mu) = (q^6, q^2(q^3 - 1), q^2(q^2 + q - 3), q^2(q^2 - 1))$.*

The distance distribution diagram of Z is



Proof: Note that the adjacency relation is symmetric, so that Z is an undirected graph. The computation of the parameters is completely straightforward. Clearly, Z has q^6 vertices. For $a, b \in W$ the maps $(u, u') \mapsto (u+a, u'+(a \times u)+b)$ are automorphisms of Z , so $\text{Aut}(Z)$ is vertex-transitive.

The $q^3 - 1$ neighbours of $(0, 0)$ are the vertices $(v, 0)$ with $v \neq 0$. The common neighbours of $(0, 0)$ and $(v, 0)$ are the vertices $(cv, 0)$ for $c \in \mathbf{F}_q, c \neq 0, 1$. Hence $a_1 = q - 2$.

The $(q^3 - 1)(q^2 - 1)$ vertices at distance 2 from $(0, 0)$ are the vertices (u, u') with $u, u' \neq 0$ and $u' \perp u$. The common neighbours of $(0, 0)$ and (u, u') are the $(v, 0)$ with $v \times u = u'$, and together with $(v, 0)$ also $(v + cu, 0)$ is a common neighbour, so $c_2 = q$. Vertices (u, u') and (v, v') , both at distance 2 from $(0, 0)$ are adjacent when $0 \neq v \perp u'$ and $v \neq u$ and $v \times u \neq u'$ and $v' = u \times v + u'$, so that $a_2 = q^2 - q - 2$.

The remaining $(q^3 - 1)(q^3 - q^2 + 1)$ vertices have distance 3 to $(0, 0)$. They are the (w, w') with $w \not\perp w'$ or $w = 0 \neq w'$. The neighbours (u, u') of (w, w') that lie at distance 2 to $(0, 0)$ satisfy $0 \neq u \perp w'$ and $(0 \neq u)u' = w \times u + w'$,

so that $c_3 = q^2 - 1$. This shows that Z is distance-regular with the claimed parameters. The spectrum follows.

The fact that the extended bipartite double is distance-regular, and has the stated intersection array, follows from [BCN], Theorem 1.11.2(vi).

The fact that Z_3 is strongly regular follows from [BCN], Proposition 4.2.17(ii) (which says that this happens when Z has eigenvalue -1). \square

For $q = 2$, the graphs here are (i) the folded 7-cube, (ii) the folded 8-cube, (iii) the halved folded 8-cube. All are distance-transitive. For $q > 2$ these graphs are not distance-transitive.

When q is a power of two, the graphs \hat{Z} have the same parameters as certain Kasami graphs, but for $q > 2$ these are nonisomorphic.

Next, a more geometric description of this graph.

Let H be a vector space of dimension 7 over the field \mathbf{F}_q , provided with a nondegenerate quadratic form. Let Γ be the graph of which the vertices are the maximal totally isotropic subspaces of H (of dimension 3), where two vertices are adjacent when their intersection has dimension 2. This graph is known as the dual polar graph of type $B_3(q)$. It is distance-regular with intersection array $\{q(q^2 + q + 1), q^2(q + 1), q^3; 1, q + 1, q^2 + q + 1\}$. (See [BCN], §9.4.)

Proposition 3.2 *Let Γ be the dual polar graph of type $B_3(q)$. Fix a vertex π_0 of Γ , and let Δ be the subgraph of Γ induced on the collection of vertices disjoint from π_0 . Then Δ is isomorphic to the graph Z of Proposition 3.1. Its extended bipartite double $\hat{\Delta}$ (or \hat{Z}) is isomorphic to the graph of Proposition 2.1.*

Proof: Let V be a vector space of dimension 8 over \mathbf{F}_q (with basis $\{e_1, \dots, e_8\}$), provided with the nondegenerate quadratic form $Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8$. The point $P = (0, 0, 0, 1, 0, 0, 0, -1)$ is nonisotropic, and P^\perp is the hyperplane H defined by $x_4 = x_8$. Restricted to H the quadratic form becomes $Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4^2$.

The D_4 -geometry on V has disjoint maximal totally isotropic subspaces $E = \langle e_1, e_2, e_3, e_4 \rangle$ and $F = \langle e_5, e_6, e_7, e_8 \rangle$. Fix E and consider the collection of all maximal totally isotropic subspaces disjoint from E . This is precisely the collection of images F_A of F under matrices $\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$, where A is alternating with zero diagonal (cf. [BCN], Proposition 9.5.1(i)). Hence, we can label the q^6 vertices $F_A \cap H$ of Δ with the q^6 matrices A .

Two vertices are adjacent when they have a line in common, that is, when they are the intersections with H of maximal totally isotropic subspaces in V , disjoint from E , that meet in a line contained in H . Let

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

Then $\det A = (af - be + cd)^2$, and if $\det A = 0$ but $A \neq 0$, then $\ker A$ has dimension 2, and is spanned by the four vectors $(0, f, -e, d)^\top$, $(-f, 0, c, -b)^\top$, $(e, -c, 0, a)^\top$, $(-d, b, -a, 0)^\top$. Writing the condition that matrices A and A' belong to adjacent vertices we find the description of Proposition 3.1 if we take $u = (c, e, f)$ and $u' = (-d, b, -a)$. \square

4 History

In 1991 the second author constructed the graphs from Section 2 and the first author those from Section 3. Both were mentioned on the web page [ac], but not published thus far. These graphs have been called the Pasechnik graphs and the Brouwer-Pasechnik graphs, respectively, by on-line servers.

References

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