D. S. Asche (cf. [1], Example 5.19) gave a construction for a system of 72 lines in $\mathbb{R}^{19}$ with mutual angles $\arccos(1/5)$. F. Szöllösi [2] discovered that Asche's system contains a subsystem of 54 lines in $\mathbb{R}^{18}$ with mutual angles $\arccos(1/5)$.

The construction in [2] is very explicit. Here we give precisely the same construction but formulated without choosing an explicit Golay code, and an explicit vector $m$.

1 Asche's construction

Let $(X, B)$ be a Steiner system $S(5, 8, 24)$. (Thus, $|X| = 24$, $|B| = 759$, members of $B$ have size 8, and each subset of size 5 of $X$ is contained in a unique member of $B$.) Call the elements of $X$ points. Call the members of $B$ octads. Octads meet in 8, 4, 2, or 0 points.

Let $B_1, B_2 \in B$ be two octads that meet in 2 points. Let $p$ be a point outside $B_1 \cup B_2$, and let $B_1 \cap B_2 = \{q, r\}$. Let $A$ be the set of 72 octads that contain $p$ but not $q$, $r$ and meet each of $B_1$ and $B_2$ in 2 points. All choices made are unique up to isomorphism, and the resulting set $A$ (for Asche) is a single orbit of the subgroup of size 144 of $M_{24}$ that stabilizes this situation.

For each subset $S$ of $X$, let $\chi_S \in \mathbb{R}^X$ be its characteristic vector (so that $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise). With the usual inner product this means that $(\chi_S, \chi_T) = |S \cap T|$. Put $u := \chi_p + \frac{1}{4} \chi_X$. Now the 72 unit vectors $(\chi_B - u)/\sqrt{5}$ have mutual inner products $\pm 1/5$, so that the system $\Phi$ of 72 lines they span is equiangular. The five conditions $p \in B$, $q \notin B$, $r \notin B$, $|B \cap B_1| = 2$, $|B \cap B_2| = 2$ force $\langle \Phi \rangle$ to have dimension 19.

2 Szöllösi's construction

Continuing the notation from the previous section, let $B_3$ be an octad containing $p, q, r$ such that $|B_3 \cap B_1| = 4$ and $|B_3 \cap B_2| = 2$. Let $s$ be a point other than $q, r$ in $B_3 \cap B_1$. Let $S$ (for Szöllösi) be the subset of size 54 of $A$ consisting of the octads $B$ such that $|B \cap B_3| = 4$ if $s \in B$ and $|B \cap B_3| = 2$ otherwise. All choices made are unique up to isomorphism.

The condition given says that $\chi_B - u$ is orthogonal to $3(\chi_{B_3} - 2\chi_{\{s\}}) + 2\chi_p - 4\chi_{\{q, r\}}$, so that the resulting subsystem $\Psi$ of $\Phi$ spans a subspace of dimension 18.
References
