Some locally Kneser graphs

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Abstract

The Kneser graph $K(n, d)$ is the graph on the d-subsets of an n-set, adjacent when disjoint. Clearly, $K(n+d, d)$ is locally $K(n, d)$. Hall showed for $n \geq 3d + 1$ that there are no further examples. Here we give other examples of locally $K(n, d)$ graphs for $n = 3d$, and some further sporadic examples. It follows that Hall's bound is best possible.

1 Locally something graphs

A graph Γ is called *locally* Δ when for each vertex of Γ the subgraph induced on the set of its neighbors is isomorphic to Δ. The Trahtenbrot-Zykov problem [31] asks whether given a finite graph Δ there exists a graph Γ that is locally Δ. In general, this question is undecidable (Bulitko [14]). It is unknown whether the problem restricted to finite Γ is also undecidable.

More generally, one wants to classify all such graphs Γ. Hall [19] determines the possible Γ for all graphs Δ on at most 6 vertices. For some Δ a graph Γ that is locally Δ is necessarily infinite.

Weetman [28, 29] constructs infinite locally Δ graphs for Δ of girth at least 6, and proves a diameter bound in certain other cases.

There is a large literature, see e.g. [1–15, 18–30].

2 Locally Kneser graphs

The *Kneser graph* $K(n, d)$ (where $0 \leq d \leq n$) is the graph on the d-subsets of an n-set, adjacent when disjoint.

Hall [20] shows that for $n \geq 3d + 1$ any connected locally $K(n, d)$ graph is isomorpic to $K(n + d, d)$, and wonders whether this bound can be improved.

In fact there are further examples for $n = 3d$: The graph on the 2^{n-1} even weight binary vectors of length n, adjacent when their difference has weight $2d$ is locally $K(n, d)$ and different from $K(n + d, d)$ (for $d > 1$). It follows that Hall's bound is best possible.

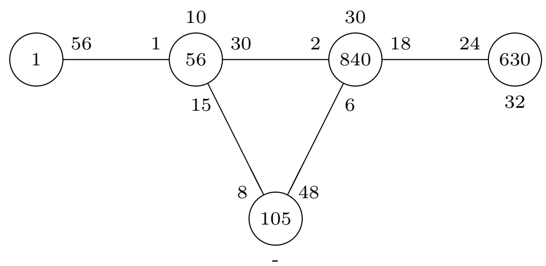
For $n = 2d + 1$, $d \geq 3$, the graph $K(n, d)$ has girth 6, so that there exist infinite locally $K(n, d)$ graphs by Weetman [28].

There are three locally $K(5, 2)$ graphs, on 21, 63, and 65 vertices (Hall [18]). Note that $K(5, 2)$ is the Petersen graph.

There are three locally $K(6,2)$ graphs, on 28, 32, and 36 vertices (Buekenhout & Hubaut [10]). Note that $K(6,2)$ is the collinearity graph of the generalized quadrangle of order 2.

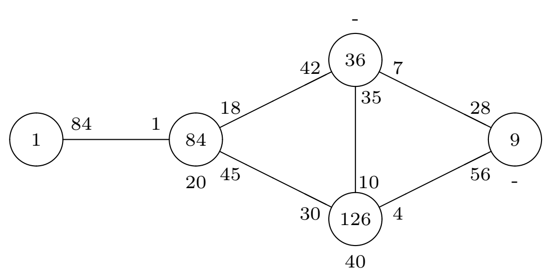
As we saw, there are infinite locally $K(7,3)$ graphs. (Are there also finite examples?)

The graph on the elliptic lines in the $O_8^-(2)$ geometry, adjacent when orthogonal, is locally $K(8,3)$. The automorphism group is $O_8^-(2){:}2$, with point stabilizer $S_3 \times S_8$. Diagram:



See also [16].

As we saw, the graph on the 256 binary even weight vectors of length 9, adjacent when they have distance 6, is locally $K(9,3)$. Diagram:



It can be shown using the arguments from [29] that a locally $K(9,3)$ graph is necessarily finite. (Is the same true for all locally $K(3d,d)$ graphs?)

The bimonster $G = M \text{ wr } 2$ (where M is the monster) contains a S_5-subgroup S whose centralizer is a subgroup S_{12} in which a 7-point stabilizer is conjugate to S, see [17]. Let Γ be the graph on the S_5-subgroups of G conjugate to S, adjacent when they commute. Then Γ is locally $K(12,5)$.

3 Locally $\lambda = 1$ graphs

Let Δ be a graph in which every edge is in a unique triangle (so that Δ is the collinearity graph of a partial linear space with lines of size 3). The Kneser graphs $K(3d,d)$ are examples of such graphs. We study locally Δ graphs. The special case where Δ is the line graph of the Petersen graph was studied in [6].

Given a partial linear space (X,L) with lines of size 3, let G be the group

$$G = \langle X \mid x^2 = 1 = xyz \text{ for all } x \in X \text{ and } \{x,y,z\} \in L \rangle.$$

Suppose A, B are two disjoint hyperplane complements in (X,L), so that each line meets A and B in 0 or 2 points. Then the map that sends the elements of A,

B, $X \setminus (A \cup B)$ to a, b, and 1, respectively, is a map from G onto the infinite group $\langle a, b \mid a^2 = b^2 = 1 \rangle$, so that G is infinite.

Given a subgroup H of G, let $\Gamma = \Gamma(G, H, X)$ be the graph that has as vertices the cosets gH for $g \in G$, and adjacencies $g_1 H \sim g_2 H$ when $g_2^{-1} g_1 \in HXH$. Assume that H is normal in G. Now the neighbours of gH are the vertices gxH for $x \in X$. The group G acts vertex transitively on Γ. The local graph induced on the set of neighbours of the vertex H of Γ has vertex set $\{xH \mid x \in X\}$, and if $\{x, y, z\}$ is a line, then $xy = z$ in G, so that $xH \sim yH$. It follows that $x \mapsto xH$ is a homomorphism from the collinearity graph Δ of (X, L) onto the Γ-neighbourhood of H.

Is this map injective? Suppose H is contained in the commutator subgroup G' of G. Then $\Gamma(G, H, X)$ has quotient $\Gamma(G, G', X)$ and the latter can be identified with the Cayley graph with difference set X in the $\mathbb{F}_2$-vector space $\langle X \mid x + y + z = 0 \text{ for } \{x, y, z\} \in L \rangle$. If N is the point-line incidence matrix of (X, L), then this is $\langle X \rangle / N \langle L \rangle$, where cosets at Hamming distance 1 are adjacent. Two points remain distinct in the quotient if the column space of N does not contain vectors of weight 2. No additional adjacencies are introduced if the columns of N are the only vectors of weight 3 in the column space of N.

For these latter two conditions to hold, it suffices that for any two distinct points, and for any three pairwise noncollinear points, (X, L) has a geometric hyperplane missing precisely one of these points.

If these conditions hold, Γ is locally Δ.

Examples

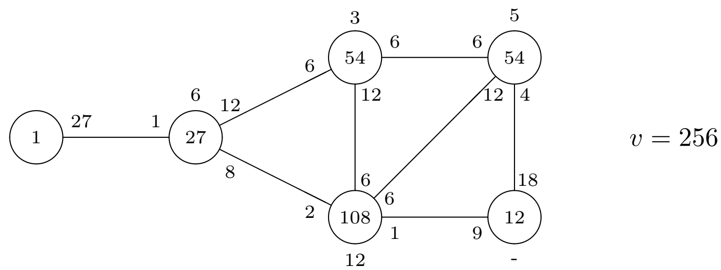
We construct $\Gamma = \Gamma(G, G', X)$ for a number of spaces (X, L) with lines of size 3. Note that $V = G/G'$ is a binary vector space. In all cases except d), the graph Γ is locally Δ, where Δ is the collinearity graph of (X, L).

	Δ	parameters	$\|G\|$	dim V	v	k	d	rk	Aut Γ
a)	$GQ(2,1)$	srg(9, 4, 1, 2)	16	4	16	9	2	3	$[2^7.3^2]$
b)	$GQ(2,2)$	srg(15, 6, 1, 3)	32	5	32	15	3	4	$2^5{:}S_6$
c)	$GQ(2,4)$	srg(27, 10, 1, 5)	64	6	64	27	2	3	$2^6{:}O_6^-(2)$
d)	$VO_4^-(3)$	srg(81, 20, 1, 6)	1	0	-				
e)	$L(K(5,2))$	$\{4, 2, 1; 1, 1, 4\}$	∞	6	64	15	3	6	$2^6{:}S_5$
f)	$GH(2,1)$	$\{4, 2, 2; 1, 1, 2\}$	∞	8	256	21	3	7	$2^8{:}PGL(3,2)$
g)	$GH(2,2)$	$\{6, 4, 4; 1, 1, 1\}$	∞	14	16384	63	4	15	$2^{14}{:}G_2(2)$
g′)	$GH(2,2)$	$\{6, 4, 4; 1, 1, 1\}$	∞	14	16384	63	6	26	$2^{14}{:}G_2(2)$
h)	3^3	$\{6, 4, 2; 1, 2, 3\}$	512	8	256	27	3	6	$[2^{12}.3^4]$
i)	3^4	$\{8, 6, 4, 2; 1, 2, 3, 4\}$		16	65536	81	6	30	$[2^{23}.3^5]$
j)	$GO(2,1)$	$\{4, 2, 2, 2; 1, 1, 1, 2\}$	∞	16	65536	45	6	93	$2^{16}{:}M_{10}$
k)	$3S_6$	$\{6, 4, 2, 1; 1, 1, 4, 6\}$	∞	11	2048	45	5	16	$2^5{:}(2^6{:}3.S_6)$
l)	$K(9,3)$	$(v,k)_\Delta = (84, 20)$	256	8	256	84	3	5	$2^8{:}S_9$
m)	$K(12,4)$	$(v,k)_\Delta = (495, 70)$	2048	11	2048	495	3	7	$2^{11}{:}S_{12}$

We give the parameters for a strongly regular graph as srg(v, k, λ, μ), and for a distance-regular graph of diameter at least 3 as $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ (cf. [5]). In cases a), b), the graphs Δ are 3^2 and $K(6,2)$. In cases a), b), and c), the graphs Γ are $VO_4^+(2)$, $T\Delta$, and $VO_6^-(2)$ (with notation as in [7]).

Cases g) and g′) are the dual Cayley and the Cayley generalized hexagon, respectively. For the former Γ has diameter 4 and Aut Γ has trivial center, for the latter Γ is antipodal of diameter 6 and Aut Γ has a center of order 2 that interchanges antipodes.

In case h) the diagram is



Cases b), l), and m) suggest that for $K(3d,d)$ the group G is elementary abelian of order 2^{3d-1}, and this is indeed easy to prove.[1] Thus, for each d this approach yields only a single graph Γ that is locally $K(3d,d)$.

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[1] Denote multiplication in G by $*$, and let juxtaposition denote disjoint union in the underlying $3d$-set Z. We show $A*B = B*A$. Let $A = ER$, $B = ES$, where $E = A \cap B$. Let $Z = EFGRST$, with $|E| = |F| = |G| = e$ and $|R| = |S| = |T| = d - e$. Then $A*B = ER*ES = FS*FR = GR*GS = ES*ER = B*A$, where $ER*ES = FS*FR$ since $ER*ES = FS*GT*GT*FR = FS*FR$, and similarly for the other two equalities.

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