The invariants of the binary nonic

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Abstract

We consider the algebra of invariants of binary forms of degree 9 with complex coefficients, find the 92 basic invariants, give an explicit system of parameters and show the existence of four more systems of parameters with different sets of degrees.

1 Introduction

Invariants

Let $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ denote the algebra of invariants of binary forms (forms in two variables) of degree n with complex coefficients. This algebra was extensively studied in the nineteenth century, and for $n \leq 6$ the structure was clear and a finite basis was known. While Cayley (1856)[1] states that for $n = 7$ there is no such finite basis, Gordan (1868) proved that $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ has a finite basis for all n. After initial work by von Gall (1880, 1888), the degrees of the basic invariants in the cases $n = 7$ and $n = 8$ were found by Dixmier & Lazard (1986) and Shioda (1967), respectively. Here we consider the case $n = 9$, and show that $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ is generated by 92 basic invariants, and give the degrees (Proposition 3.1). Earlier work on the case $n = 9$ was done by Sylvester & Franklin (1879) and by Cröni (2002).

Systems of parameters

A (homogeneous) *system of parameters* for a graded algebra A is an algebraically independent set S of homogeneous elements of A such that A is module-finite over the subalgebra generated by the set S. Hilbert (1893) showed the existence of a system of parameters for algebras of invariants, cf. Proposition 5.1 below.

In the case $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ considered here, Dixmier (1985) proved

Proposition 1.1. *$\mathcal{O}(V_9)^{\mathrm{SL}_2}$ has a homogeneous system of parameters of degrees 4, 8, 10, 12, 12, 14, 16.*

but was unable to give an explicit such system. Here we find an explicit system of parameters for $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ (Theorem 4.1), and show the existence of systems of parameters for certain further sequences of degrees (Proposition 7.2).

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[1]See references at the end of this note.

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2 Invariants and Poincaré series

Let $V_n = \mathbb{C}[x, y]_n$ be the SL_2-module of binary forms (homogeneous polynomials in x and y) of degree n, on which SL_2 acts via

$$g \cdot f(v) = f(g^{-1}v),$$

for $g \in \mathrm{SL}_2$, $f \in \mathbb{C}[x, y]$ and $v \in \mathbb{C}^2$. The coordinate ring of V_n, denoted by $\mathcal{O}(V_n)$, is isomorphic to the polynomial ring $\mathbb{C}[a_0, \ldots, a_n]$. The group SL_2 acts on the coordinate ring $\mathcal{O}(V_n)$ via the action

$$g \cdot j(f) = j(g^{-1} \cdot f),$$

for $g \in \mathrm{SL}_2$, $j \in \mathcal{O}(V_n)$ and $f \in V_n$. An *invariant* of V_n is a homogeneous element $j \in \mathcal{O}(V_n)$ such that $g \cdot j = j$ for all $g \in \mathrm{SL}_2$. The set of elements of $\mathcal{O}(V_n)$ invariant under the action of SL_2 forms the *ring of invariants* $I := \mathcal{O}(V_n)^{\mathrm{SL}_2}$.

This ring of invariants I is graded by degree, so that $I = \oplus_m I_m$, where I_m is the subspace of I consisting of the invariants that are homogeneous of degree m. The Poincaré series (or Hilbert series) of I is the series $P(t) = \sum_m \dim_{\mathbb{C}}(I_m)t^m$. Already Cayley and Sylvester ([3, 19]) knew how to compute this Poincaré series. For a modern account, see, e.g., Springer [18]. In our case ($n = 9$) the series is given by

$$P(t) = \frac{a(t)}{(1-t^4)(1-t^8)(1-t^{10})(1-t^{12})^2(1-t^{14})(1-t^{16})}$$

with

$$\begin{aligned} a(t) = 1 &+ t^4 + 5t^8 + 4t^{10} + 17t^{12} + 20t^{14} + 47t^{16} + 61t^{18} + 97t^{20} + \\ &120t^{22} + 165t^{24} + 189t^{26} + 223t^{28} + 241t^{30} + 254t^{32} + 254t^{34} + \\ &241t^{36} + 223t^{38} + 189t^{40} + 165t^{42} + 120t^{44} + 97t^{46} + 61t^{48} + \\ &47t^{50} + 20t^{52} + 17t^{54} + 4t^{56} + 5t^{58} + t^{62} + t^{66}, \end{aligned}$$

so that

$$\begin{aligned} P(t) = 1 &+ 2t^4 + 8t^8 + 5t^{10} + 28t^{12} + 27t^{14} + 84t^{16} + 99t^{18} + 217t^{20} + \\ &273t^{22} + 506t^{24} + 647t^{26} + 1066t^{28} + 1367t^{30} + 2082t^{32} + 2649t^{34} + \\ &3811t^{36} + 4796t^{38} + 6612t^{40} + 8228t^{42} + 10960t^{44} + 13483t^{46} + \\ &17487t^{48} + 21274t^{50} + 26979t^{52} + 32490t^{54} + 40443t^{56} + 48242t^{58} + \\ &59107t^{60} + 69885t^{62} + 84470t^{64} + 99074t^{66} + \ldots \end{aligned}$$

3 The basic invariants

A minimal set of generators for the algebra I is called a set of 'basic invariants'. Such a set is not unique, but whenever there is a reference to a basic invariant we mean a member of such a set, fixed in that context. Let J_m be the subspace of I_m generated by products of invariants of smaller degree, that is, in $\bigcup_{j<m} I_j$. The number of basic invariants of degree m is $d_m := \dim_{\mathbb{C}}(I_m/J_m)$.

Proposition 3.1. *The algebra I of invariants for the binary nonic (form of degree 9) is generated by 92 invariants. The nonzero numbers d_m of basic invariants of degree m are*

m	4	8	10	12	14	16	18	20	22
d_m	2	5	5	14	17	21	25	2	1

Finding a basis for the invariants is a simple but boring procedure: For each degree m, multiply invariants of lower degrees to see what part of I_m is known already. The Poincaré series tells us how large I_m is, and if the known invariants do not yet span it, one finds in some way some more invariants, until they do span.

This procedure terminates. Gordan [11] shows that the algebra I is generated by finitely many of its elements. Better, we know when to stop. By Proposition 1.1, I has system of parameters of degrees 4, 8, 10, 12, 12, 14, 16. Let H be the ideal in I generated by such a system of parameters. Now the Poincaré series tells us that if $a(t) = \sum a_i t^i$ then $\dim_{\mathbb{C}}(I_i/(I_i \cap H)) = a_i$, and, in particular, that $I_i \subseteq H$ for $i > 66$. This means that $d_m = 0$ for $m > 66$. We followed this procedure, and found the stated values for d_m. These values agree with those given in [5] for $m \leq 20$. The existence of a basic invariant of degree 22 was new.

This 'finding more invariants in some way' was done by generating random bracket monomials[2]. Explicit bracket monomials for a set of basic invariants are listed in [1]. Checking whether the invariants known span I_m required computing a basis for vector spaces of dimension at most $\dim_{\mathbb{C}}(I_{66}) = 99074$. That is large but doable. The entire computation can be done in less than a month.

4 A system of parameters for $\mathcal{O}(V_9)^{\mathrm{SL}_2}$

Dixmier [7] proved that the invariant ring of V_9 has a system of parameters of degrees 4, 8, 10, 12, 12, 14, and 16. We compute an explicit system of parameters of $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ having these degrees.

A *covariant of order m and degree d* of V_n is an SL_2-equivariant homogeneous polynomial map $\phi : V_n \to V_m$ of degree d such that $\phi(g \cdot f) = g \cdot \phi(f)$ for all $g \in \mathrm{SL}_2$ and $f \in V_n$. The invariants of V_n are the covariants of order 0. The identity map is a covariant of order n and degree 1. Customarily, one indicates

[2]For the classical concept of bracket monomial, cf. [15].

such a covariant ϕ by giving its image of a generic element $f \in V_n$. (In particular, the identity map is noted f.) Let $V_{m,d}$ be the space of covariants of order m and degree d.

The simplest examples of covariants are obtained using *transvectants*: given $g \in V_m$ and $h \in V_n$ the expression

$$(g,h) \mapsto (g,h)_p := \frac{(m-p)!(n-p)!}{m!n!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{\partial^p g}{\partial x^{p-i} \partial y^i} \frac{\partial^p h}{\partial x^i \partial y^{p-i}}$$

defines a linear and SL_2-equivariant map $V_m \otimes V_n \to V_{m+n-2p}$, which is classically called the *p-th transvectant* (Überschiebung). We have $(g,h)_0 = gh$ and $(g,g)_{2i+1} = 0$ for all integers $i \geq 0$. These maps are the components of the Clebsch-Gordan isomorphism (for $m \geq n$)

$$V_m \otimes V_n \simeq V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{m-n}.$$

These maps induce maps $V_{m,d} \otimes V_{n,e} \to V_{m+n-2p,d+e}$.

For $f \in V_9$, consider the following covariants

$$\begin{aligned} l &= (f,f)_8 \in V_{2,2}, & r &= (q,f)_6 \in V_{3,3}, \\ q &= (f,f)_6 \in V_{6,2}, & p &= (f,l)_2 \in V_{7,3}, \\ u &= (f,f)_2 \in V_{14,2}, & k_q &= (q,q)_4 \in V_{4,4}, \end{aligned}$$

and invariants (the suffix indicates the degree)

$$\begin{aligned} j_4 &= (l,l)_2, & B_8 &= (q,r^2)_6, \\ j_{12} &= ((k_q,k_q)_2,k_q)_4, & B_{12} &= ((p,p)_4,l^3)_6, \\ j_{14} &= (q,(r^3,r)_3)_6, & D_{10} &= ((((u,u)_{10},f)_6,(q,f)_2)_5,q)_6, \\ j_{16} &= ((p,p)_2,l^5)_{10}. \end{aligned}$$

Theorem 4.1. *The seven invariants j_4, B_8, D_{10}, B_{12}, j_{12}, j_{14}, j_{16} form a homogeneous system of parameters for the ring $\mathcal{O}(V_9)^{SL_2}$ of invariants of the binary nonic.*

This is proved by invoking Hilbert's characterization of homogeneous systems of parameters as sets that define the nullcone.

5 The nullcone

The *nullcone* of V_n, denoted $\mathcal{N}(V_n)$, is the set of binary forms of degree n on which all invariants vanish. It turns out that this is precisely the set of binary forms of degree n with a root of multiplicity $> \frac{n}{2}$. The elements of $\mathcal{N}(V_n)$ are called *nullforms*. The nullcone $\mathcal{N}(V_n \oplus V_m)$ is the set of pairs $(g,h) \in V_n \oplus V_m$ such that g and h have a common root of multiplicity $> \frac{n}{2}$ in g and of multiplicity $> \frac{m}{2}$ in h.

We have the following result, due to Hilbert [13], formulated for the particular case of binary forms:

Proposition 5.1. *For $n \geq 3$, consider $i_1, \ldots, i_{n-2} \in \mathcal{O}(V_n)^{\mathrm{SL}_2}$ homogeneous non-constant invariants of V_n. The following two conditions are equivalent:*

(i) $\mathcal{N}(V_n) = \mathcal{V}(i_1, \ldots, i_{n-2})$,

(ii) $\{i_1, \ldots, i_{n-2}\}$ *is a homogeneous system of parameters of* $\mathcal{O}(V_n)^{\mathrm{SL}_2}$.

In other words, if we find homogeneous invariants $i_1, \ldots, i_{n-2}$ such that $\mathcal{N}(V_n) = \mathcal{V}(i_1, \ldots, i_{n-2})$, it follows that $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ is a finitely generated module over $\mathbb{C}[i_1, \ldots, i_{n-2}]$. But invariant rings of binary forms are Cohen-Macaulay ([14]), which implies that $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ is a free $\mathbb{C}[i_1, \ldots, i_{n-2}]$-module. Hence the description of the algebra of invariants of V_n is partly reduced to finding a system of parameters of $\mathcal{O}(V_n)^{\mathrm{SL}_2}$.

We prove the above theorem by first finding a defining set for the nullcone that is still too large, and then showing that some elements are superfluous.

We need information on the invariants of V_n for $n = 2, 3, 6, 7$:

Lemma 5.2. *The following are systems of parameters of $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ for $n = 2, 3, 6, 7$.*

(i) If $n = 2$: $(f, f)_2$ of degree 2.

(ii) If $n = 3$: $((f, f)_2, (f, f)_2)_2$ of degree 4.

(iii) If $n = 6$: $(f, f)_6$, $(k_f, k_f)_4$, $((k_f, k_f)_2, k_f)_4$, and $(m_f^2, (k_f, k_f)_2)_4$ of degrees 2, 4, 6, and 10, where $k_f = (f, f)_4$ and $m_f = (f, k_f)_4$.

(iv) If $n = 7$: $(l_f, l_f)_2$, $((p_f, p_f)_4, l_f)_2$, $((k_{qf}, k_{qf})_2, k_{qf})_4$, $((p_f, p_f)_2, l_f^3)_6$, and $(m_{qf}^2, (k_{qf}, k_{qf})_2)_4$ of degrees 4, 8, 12, 12, and 20, where $l_f = (f, f)_6$, $p_f = (f, l_f)_2$, $q_f = (f, f)_4$, $k_{qf} = (q_f, q_f)_4$, $m_{qf} = (q_f, k_{qf})_4$.

Proof. This is classical for $n = 2, 3, 6$, see, e.g., [4, 12, 16]. Systems of parameters for $n = 7$ were given by Dixmier [6] and Bedratyuk [2]. The above system was constructed by the second author (unpublished). That it is a system of parameters can be easily verified using the methods of this section. □

Lemma 5.3. (Weyman [21]) *Let $f \in V_d$. If $d > 4k - 4$ and all $(f, f)_{2k}$, $(f, f)_{2k+2}$, ... vanish, then f has a root of multiplicity $d - k + 1$. If $d = 4k - 4$ and $((f, f)_{2k-2}, f)_d$, $(f, f)_{2k}$, $(f, f)_{2k+2}$, ... vanish, then f has a root of multiplicity $d - k + 1$.* □

Lemma 5.4. *Let $f \in V_9$ and consider its covariants $l = (f, f)_8$, $q = (f, f)_6$, $p = (f, l)_2$, and $r = (f, q)_6$.*

(i) If $l \neq 0$ and $(l, p) \in \mathcal{N}(V_2 \oplus V_7)$, then f has a root of multiplicity 5.

(ii) If $l = 0$, $q \neq 0$ and $(q, r) \in \mathcal{N}(V_6 \oplus V_3)$ then f has a root of multiplicity 6.

(iii) If $l = q = 0$, then f has a root of multiplicity 7.

Proof. Let $f = \sum_{i=0}^{9} \binom{9}{i} a_i x^{9-i} y^i$.

(i). From $(l, p) \in \mathcal{N}(V_2 \oplus V_7)$ it follows that both l and p are nullforms and have a common root of multiplicity 2 in l and 4 in p. Without loss of generality we suppose $l = x^2$. Then:

$$p = (f, x^2)_2 = \frac{1}{72} \sum_{i=2}^{9} \binom{9}{i} i(i-1) a_i x^{9-i} y^{i-2},$$

and x^4 must divide p, which implies $a_6 = a_7 = a_8 = a_9 = 0$. Now

$$l = (f, f)_8 = 70a_5^2 y^2 + 28a_4 a_5 xy + (70a_4^2 - 112a_3 a_5)x^2,$$

and as we suppose $l = x^2$ we also obtain $a_5 = 0$ and then it follows that $x^5 \mid f$, so f will have a root of multiplicity 5.

(ii). From $(q, r) \in \mathcal{N}(V_6 \oplus V_3)$ it follows that both q and r are nullforms and have a common root of multiplicity 4 in q and 2 in r. Without loss of generality we consider the following 3 cases: $q = x^6$, $q = x^5 y$, and $q = x^4 y(x + y)$.
Case 1: $q = x^6$. Then

$$r = (f, x^6)_6 = a_9 y^3 + 3a_8 xy^2 + 3a_7 x^2 y + a_6 x^3,$$

and x^2 must divide r. We obtain $a_9 = a_8 = 0$ and substitute that in q and l:

$$\begin{aligned}
q = (f, f)_6 = &(-20a_6^2 + 30a_5 a_7)y^6 + (-30a_5 a_6 + 54a_4 a_7)xy^5 + \\
&(-90a_5^2 + 114a_4 a_6 - 12a_3 a_7)x^2 y^4 + \\
&(-72a_4 a_5 + 124a_3 a_6 - 60a_2 a_7)x^3 y^3 + \\
&(-90a_4^2 + 114a_3 a_5 - 12a_2 a_6 - 18a_1 a_7)x^4 y^2 + \\
&(-30a_3 a_4 + 54a_2 a_5 - 30a_1 a_6 + 6a_0 a_7)x^5 y + \\
&(-20a_3^2 + 30a_2 a_4 - 12a_1 a_5 + 2a_0 a_6)x^6, \\
l = (f, f)_8 = &(70a_5^2 - 112a_4 a_6 + 56a_3 a_7)y^2 + \\
&(28a_4 a_5 - 56a_3 a_6 + 40a_2 a_7)xy + \\
&(70a_4^2 - 112a_3 a_5 + 56a_2 a_6 - 16a_1 a_7)x^2.
\end{aligned}$$

Since we suppose $q = x^6$ and $l = 0$, the coefficients of $x^i y^{6-i}$ in q and of $x^j y^{2-j}$ in l are 0 for $0 \leq i \leq 5$ and $0 \leq j \leq 2$.

If $a_7 = 0$ then it follows that $a_6 = a_5 = a_4 = 0$ and then $x^6 \mid f$, so f will have a root of multiplicity 6. If $a_7 \neq 0$ then

$$\begin{aligned}
a_5 &= \frac{2a_6^2}{3a_7}, \quad a_4 = \frac{10a_6^3}{27a_7^2}, \quad a_3 = \frac{5a_6^4}{27a_7^3}, \\
a_2 &= \frac{7a_6^5}{81a_7^4}, \quad a_1 = \frac{28a_6^6}{729a_7^5}, \quad a_0 = \frac{4a_6^7}{243a_7^6},
\end{aligned}$$

but then we have $q = 0$, contrary to the assumption.

Case 2: $q = x^5y$. Then

$$r = (f, x^5y)_6 = -a_8y^3 - 3a_7xy^2 - 3a_6x^2y - a_5x^3$$

and x^2 must divide r. We obtain $a_8 = a_7 = 0$ and substitute this in q and l:

$$\begin{aligned} q = (f,f)_6 = &(-20a_6^2 + 2a_3a_9)y^6 + (-30a_5a_6 + 6a_2a_9)xy^5 + \\ &(-90a_5^2 + 114a_4a_6 + 6a_1a_9)x^2y^4 + \cdots + \\ &(-90a_4^2 + 114a_3a_5 - 12a_2a_6)x^4y^2 + \\ &(-30a_3a_4 + 54a_2a_5 - 30a_1a_6)x^5y + \cdots \\ l = (f,f)_8 = &(70a_5^2 - 112a_4a_6 + 2a_1a_9)y^2 + \cdots \end{aligned}$$

Since we supposed $q = x^5y$ and $l = 0$, the coefficient c of y^2 in l, and the coefficients d_i of x^iy^{6-i} in q vanish for $0 \le i \le 4$, while $d_5 \neq 0$. Now

$$5d_5a_9 = -75a_4d_0 + 45a_5d_1 - a_6(9c + 22d_2) = 0$$

so that $a_9 = 0$, and then also $a_6 = a_5 = a_4 = 0$, $d_5 = 0$, contradicting $d_5 \neq 0$.

Case 3: $q = x^4y(x + y)$. Then:

$$r = (f, x^4y(x+y))_6 = (a_7 - a_8)y^3 + 3(a_6 - a_7)xy^2 + 3(a_5 - a_6)x^2y + (a_4 - a_5)x^3$$

and x^2 must divide r. We obtain $a_8 = a_7 = a_6$ which we replace in q and l:

$$\begin{aligned} q = (f,f)_6 = &-2(6a_4a_6 - 15a_5a_6 + 10a_6^2 - a_3a_9)y^6 - \\ &-6(5a_3a_6 - 9a_4a_6 + 5a_5a_6 - a_2a_9)xy^5 - \\ &-6(15a_5^2 + 3a_2a_6 + 2a_3a_6 - 19a_4a_6 - a_1a_9)x^2y^4 - \\ &-2(36a_4a_5 - 3a_1a_6 + 30a_2a_6 - 62a_3a_6 - a_0a_9)x^3y^3 - \\ &-6(15a_4^2 - 19a_3a_5 - a_0a_6 + 3a_1a_6 + 2a_2a_6)x^4y^2 - \\ &-6(5a_3a_4 - 9a_2a_5 - a_0a_6 + 5a_1a_6)x^5y - \\ &-2(10a_3^2 - 15a_2a_4 + 6a_1a_5 - a_0a_6)x^6, \\ l = (f,f)_8 = &2(35a_5^2 - 8a_2a_6 + 28a_3a_6 - 56a_4a_6 + a_1a_9)y^2 + \\ &2(14a_4a_5 - 7a_1a_6 + 20a_2a_6 - 28a_3a_6 + a_0a_9)xy + \\ &2(35a_4^2 - 56a_3a_5 + a_0a_6 - 8a_1a_6 + 28a_2a_6)x^2. \end{aligned}$$

As we supposed $q = x^4y(x+y)$ and $l = 0$, the coefficients of y^6, xy^5, x^2y^4, x^3y^3, x^6 in q and all coefficients of l must vanish. We denote by I the ideal generated by these coefficients. Also, we denote by p_1, p_2 the coefficients of x^4y^2 and x^5y in q:

$$\begin{aligned} p_1 &= 15a_4^2 - 19a_3a_5 - a_0a_6 + 3a_1a_6 + 2a_2a_6, \\ p_2 &= 5a_3a_4 - 9a_2a_5 - a_0a_6 + 5a_1a_6. \end{aligned}$$

A Gröbner basis computation shows that p_1^4, $p_2^2 \in I$ so that p_1 and p_2 vanish, contradicting the assumption $q = x^4y(x + y)$.

(iii). This is a consequence of Lemma 5.3. □

Lemma 5.5. *Let $g \in V_2$ and $h \in V_7$ two non-zero binary forms. If both g and h are nullforms and if*

$$((h,h)_6, g)_2 = ((h,h)_4, g^3)_6 = ((h,h)_2, g^5)_{10} = (h^2, g^7)_{14} = 0,$$

then $(g,h) \in \mathcal{N}(V_2 \oplus V_7)$.

Proof. Suppose that $(g,h) \notin \mathcal{N}(V_2 \oplus V_7)$. This means that g and h have no common root which has multiplicity 2 in g and multiplicity 4 in h. Without loss of generality we suppose

$$g = x^2,$$
$$h = y^4(b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3).$$

We have then

$$0 = ((h,h)_6, g)_2 = -\frac{4}{245}b_1^2,$$
$$0 = ((h,h)_4, g^3)_6 = \frac{2}{735}(5b_2^2 - 12b_1b_3),$$
$$0 = ((h,h)_2, g^5)_{10} = -\frac{2}{147}(3b_3^2 - 7b_2b_4),$$
$$0 = (h^2, g^7)_{14} = b_4^2$$

and it follows that $b_1 = b_2 = b_3 = b_4 = 0$, which implies $h = 0$. This contradicts the assumption that $h \neq 0$. □

Lemma 5.6. *Let $g \in V_6$, $h \in V_3$ be two non-zero binary forms. If both g and h are nullforms and if*

$$((g^2,g)_6, h^2)_6 = (((g,g)_2, g)_1, h^4)_{12} = (g, h^2)_6 = (g, (h,h)_2^3)_6 = (g, (h^3,h)_3)_6 = 0$$

then $(g,h) \in \mathcal{N}(V_6 \oplus V_3)$.

Proof. Suppose that $(g,h) \notin \mathcal{N}(V_6 \oplus V_3)$. This means that g and h have no common root which has multiplicity 4 in g and multiplicity 2 in h. Without loss of generality we consider two cases:

$$g = x^4(b_1x^2 + b_2xy + b_3y^2),$$
$$h = y^3$$

and

$$g = x^4(b_1x^2 + b_2xy + b_3y^2),$$
$$h = xy^2.$$

Case 1: $h = y^3$. Then we have:

$$0 = ((g^2,g)_6, h^2)_6 = \frac{1}{495}b_3^3,$$
$$0 = (((g,g)_2, g)_1, h^4)_{12} = -\frac{1}{540}b_2(5b_2^2 - 18b_1b_3),$$
$$0 = (g, h^2)_6 = b_1$$

and it follows that $b_1 = b_2 = b_3 = 0$, which implies $g = 0$, contradicting the assumption $g \neq 0$.
Case 2: $h = xy^2$. Then we have:

$$\begin{aligned} 0 &= (g, h^2)_6 = \frac{1}{15} b_3, \\ 0 &= (g, (h,h)_2^3)_6 = -\frac{8}{729} b_1, \\ 0 &= (g, (h^3, h)_3)_6 = \frac{1}{84} b_2 \end{aligned}$$

and it follows that $b_1 = b_2 = b_3 = 0$, which implies $g = 0$, contradicting the assumption $g \neq 0$. □

Proof of Theorem 4.1
We consider the following covariants of V_9:

$$\begin{aligned} l_p &= (p,p)_6 \ \in V_{2,6}, & q_p &= (p,p)_4 \ \in V_{6,6}, \\ p_p &= (p,l_p)_2 \ \in V_{5,9}, & k_{qp} &= (q_p,q_p)_4 \ \in V_{4,12}, \\ k_q &= (q,q)_4 \ \in V_{4,4}, & m_{qp} &= (q_p,k_{qp})_4 \in V_{2,18}, \\ m_q &= (q,k_q)_4 \ \in V_{2,6}, \end{aligned}$$

and the following invariants of V_9:

$$\begin{aligned} j_4 &= (l,l)_2, & A_4 &= (q,q)_6, \\ j_8 &= (k_q,k_q)_4, & A_8 &= ((p,p)_6,l)_2, \\ j_{12} &= ((k_q,k_q)_2,k_q)_4, & A_{12} &= (l_p,l_p)_2, \\ j_{14} &= (q,(r^3,r)_3)_6, & A_{20} &= (p^2,l^7)_{14}, \\ j_{16} &= ((p,p)_2,l^5)_{10}, & A_{36} &= ((p_p,p_p)_2,l_p^3)_6, \\ j_{18} &= ((((q,q)_2,q)_1,r^4)_{12}, & B_8 &= (q,r^2)_6, \\ j_{20} &= (m_q^2,(k_q,k_q)_2)_4, & B_{12} &= ((p,p)_4,l^3)_6, \\ j_{24} &= ((p_p,p_p)_4,l_p)_2, & B_{20} &= (q,(r,r)_2^3)_6, \\ j_{36} &= ((k_{qp},k_{qp})_2,k_{qp})_4, & C_{12} &= ((r,r)_2,(r,r)_2)_2, \\ j_{60} &= (m_{qp}^2,(k_{qp},k_{qp})_2)_4, & D_{12} &= ((q^2,q)_6,r^2)_6. \end{aligned}$$

Apply Lemma 5.2 to $l \in V_2$, $r \in V_3$, $q \in V_6$ and $p \in V_7$. It follows that if $j_4 = 0$ then l is a nullform, if $C_{12} = 0$ then r is a nullform, if $A_4 = j_8 = j_{12} = j_{20} = 0$ then q is a nullform, and if $A_{12} = j_{24} = j_{36} = A_{36} = j_{60} = 0$, then p is a nullform. If we combine this information with Lemma 5.4, Lemma 5.5 and Lemma 5.6 we obtain that

$$\mathcal{N}(V_9) = \mathcal{V}(j_4, A_4, j_8, A_8, B_8, j_{12}, A_{12}, B_{12}, C_{12}, D_{12}, j_{14}, j_{16}, j_{18}, j_{20}, A_{20}, B_{20}, j_{24}, j_{36}, A_{36}, j_{60}).$$

This can be improved to the following result:

Proposition 5.7. *The nullcone $\mathcal{N}(V_9)$ is the zero set of the following invariants:*

$$\mathcal{N}(V_9) = \mathcal{V}(j_4, A_4, j_8, A_8, j_{12}, B_{12}, j_{14}, j_{16}, j_{20}, A_{20}).$$

Proof. If $j_4 = 0$ then l is a nullform.

Case 1: $l = 0$.
If $A_4 = j_8 = j_{12} = j_{20} = 0$ then q is a nullform. Without loss of generality we suppose $x^4 \mid q$. Modulo the ideal generated by the coefficients of l and the coefficients of $x^3y^3, x^2y^4, xy^5, y^6$ in q we have

$$B_8 = C_{12} = D_{12} = j_{18} = B_{20} = 0.$$

(This was an easy computation in Mathematica.) From Lemma 5.4 it follows then that if $l = 0$ and

$$A_4 = j_8 = j_{12} = j_{14} = j_{20} = 0,$$

then f is a nullform.

Case 2: $l = x^2$ (without loss of generality).
Here we have:

$$\begin{aligned}
A_{20} &= a_9^2, \\
j_{16} &= -2(a_8^2 - a_7a_9), \\
B_{12} &= 2(3a_7^2 - 4a_6a_8 + a_5a_9), \\
A_8 &= -2(10a_6^2 - 15a_5a_7 + 6a_4a_8 - a_3a_9).
\end{aligned}$$

Hence if $A_{20} = j_{16} = B_{12} = A_8 = 0$, then $a_9 = a_8 = a_7 = a_6 = 0$, and if we combine this with $l = x^2$ we get $a_5 = 0$ too, hence f is a nullform. □

But we are still not in the position to apply Proposition 5.1. For that we have to refine our result even more.

We introduce the covariant $s = (f, f)_4 \in V_{10,2}$ and the following invariants:

$$\begin{aligned}
C_8 &= ((q,q)_4, l^2)_4, \\
D_8 &= ((q,q)_4, (q,s)_6)_4, \\
j_{10} &= ((p,(f,q)_6)_3, (q,q)_4)_4, \\
A_{10} &= ((p,(f,q)_6)_3, l^2)_4, \\
B_{10} &= ((((f,q)_6,(f,s)_6)_3, (s,s)_8)_4, \\
C_{10} &= ((((s,s)_6, f)_6, (l,f)_2)_3, q)_6, \\
D_{10} &= ((((u,u)_{10}, f)_6, (q,f)_2)_5, q)_6.
\end{aligned}$$

The invariants j_8, A_8, B_8, C_8, and D_8 are linearly independent and together with j_4^2, A_4^2, A_4j_4 generate the vector space of invariants of degree 8 which is of dimension 8. (This can be seen, e.g., by a small computation in Mathematica.)

In a similar way it can be seen that the vector space of invariants of degree 10 is generated by j_{10}, A_{10}, B_{10}, C_{10}, and D_{10}.

Using invariants of degree ≤ 16 we built a list of 219 monomials of degree 20, each of them dividing one of the invariants j_4, A_4, j_8, A_8, B_8, C_8, D_8, C_{10} or D_{10}, to which we added

$$\begin{aligned} B_{20} &= ((r,r)_2^3, q)_6, \\ C_{20} &= (((r^3,r)_3,q)_4, ((f,u)_8,(f,s)_8)_3)_4. \end{aligned}$$

Let I be the ring of invariants, and I_i its i-th graded part. We evaluated the monomials at $\dim_{\mathbb{C}}(I_{20}) = 217$ random points in V_9, giving as result a matrix of (full) rank 217. Adding j_{20}, A_{20}, j_{10}^2, A_{10}^2, and B_{10}^2 to the list of monomials and repeating the evaluation step gave (of course) again matrices of rank 217. From the nullspaces of these matrices we obtained the relations

$$j_{20}, A_{20}, j_{10}^2, A_{10}^2, B_{10}^2 \in (j_4, A_4, j_8, A_8, B_8, C_8, D_8, C_{10}, D_{10})$$

(that is, B_{20} and C_{20} are not needed to span the elements mentioned).

Using invariants of degree ≤ 20 we built a list of 3561 monomials of degree 32, each of them dividing one of the invariants j_4, B_8, D_8, C_{10}, D_{10}, j_{12}, B_{12}, j_{14}, or j_{16}. We evaluated the monomials at $\dim_{\mathbb{C}}(I_{32}) = 2082$ random points in V_9, and this resulted in a matrix of rank 2082. The rank computations were made modulo 32003, but as we obtained the maximal rank, these monomials must generate I_{32}. It follows that

$$j_8, A_8, C_8, A_4 \in \sqrt{(j_4, B_8, D_8, C_{10}, D_{10}, j_{12}, B_{12}, j_{14}, j_{16})},$$

and then, combining it with Proposition 5.7, we get

$$\mathcal{N}(V_9) = \mathcal{V}(j_4, B_8, D_8, C_{10}, D_{10}, j_{12}, B_{12}, j_{14}, j_{16}).$$

In the same way one can show that

$$\mathcal{N}(V_9) = \mathcal{V}(A_4, B_8, D_8, C_{10}, D_{10}, j_{12}, B_{12}, j_{14}, j_{16}).$$

Remains to remove two elements from one of these two sets of generators. Since this did not seem easy to do by hand, we reverted to the boring approach, as follows. Let $H = (j_4, B_8, D_{10}, j_{12}, B_{12}, j_{14}, j_{16})$. We computed $\dim_{\mathbb{C}}(I_i \cap H)$ for $i \leq 60$ and found $\dim_{\mathbb{C}}(I_{60} \cap H) = 59107 = \dim_{\mathbb{C}}(I_{60})$, so that $I_{60} \subseteq H$. But then H contains powers of all invariants of degrees 4, 10, 20, so that in particular $A_4, C_{10} \in \sqrt{H}$. Now let $H' = (j_4, A_4, B_8, D_{10}, j_{12}, B_{12}, j_{14}, j_{16})$. We computed $\dim_{\mathbb{C}}(I_i \cap H')$ for $i \leq 40$ and found $\dim_{\mathbb{C}}(I_{40} \cap H') = 6612 = \dim_{\mathbb{C}}(I_{40})$, so that $I_{40} \subseteq H'$. But then H' contains powers of all invariants of degree 8, so that in particular $D_8 \in \sqrt{H'}$. But then $\sqrt{H} = \sqrt{H'} = I$. Thus,

$$\mathcal{N}(V_9) = \mathcal{V}(j_4, B_8, D_{10}, j_{12}, B_{12}, j_{14}, j_{16}),$$

and from Proposition 5.1 it follows that $\{j_4, B_8, D_{10}, j_{12}, B_{12}, j_{14}, j_{16}\}$ is a homogeneous system of parameters of I. $\square$

As a consequence of this result, the proof of Proposition 3.1 no longer requires Proposition 1.1. On the other hand, since the end of the proof of the theorem needs computer work anyway, one can avoid all discussion of the nullcone following Proposition 5.1 and show directly that $\sqrt{H} = I$. From Proposition 3.1 we learn that I is generated by invariants of degrees 4, 8, 10, 12, 14, 16, 18, 20, 22. Now one can verify that $I_m \subseteq H'$ for $36 \leq m \leq 44$ and $m = 48$, hence $\sqrt{H} = \sqrt{H'} = I$. Thus, Theorem 4.1 also follows from Dixmier [7] and computer work.

6 The degrees in a system of parameters

We give some restrictions on the set of degrees for the forms in a homogeneous system of parameters (hsop). Assume $n \geq 3$.

Lemma 6.1. *Fix integers j, t with $t > 0$. If an invariant of degree d is nonzero on a form $\sum a_i x^{n-i} y^i$ with the property that all nonzero a_i have $i \equiv j \pmod t$, then $d(n-2j)/2 \equiv 0 \pmod t$.*

Proof. For an invariant of degree d with nonzero term $\prod a_i^{m_i}$ we have $\sum m_i = d$ and $\sum i m_i = nd/2$. If $i \equiv j \pmod t$ when $a_i \neq 0$, then $nd/2 = \sum i m_i \equiv j \sum m_i = jd \pmod t$. □

Lemma 6.2. *Fix integers j, t with $t > 1$ and $0 \leq j \leq n$. Among the degrees d of a hsop, at least $\lfloor (n-j)/t \rfloor$ satisfy $d(n-2j)/2 \equiv 0 \pmod t$.*

Proof. We may suppose $0 \leq j < t$. There are $1 + \lfloor (n-j)/t \rfloor$ coefficients a_i with $i \equiv j \pmod t$, so that the subvariety of V_n defined by $a_i = 0$ for $i \not\equiv j \pmod t$ has dimension at least $\lfloor (n-j)/t \rfloor$. If this is zero, there is nothing to prove. Otherwise, adding the conditions that the elements of a hsop vanish reduces this subvariety to a subset of the nullcone. But the part of this subvariety defined by $a_i \neq 0$ for $i \equiv j \pmod t$ is disjoint from the nullcone. Indeed, consider the form $a_j x^{n-j} y^j + \cdots + a_{n-k} x^k y^{n-k}$, where $0 \leq j < t$ and $0 \leq k < t$ and $j + k \leq n - t$ and a_j, a_{n-k} are nonzero but $a_i = 0$ when $i \not\equiv j \pmod t$. The nullcone consists of the forms with a zero of multiplicity more than $n/2$, but $x = 0$ and $y = 0$ are zeros of multiplicity j and k, respectively, and if e.g. $j > n/2$, then $k \leq n - t - j < n - 2j < 0$, impossible. This means that a zero of multiplicity more than $n/2$ also is a zero of $a_j x^{n-j-k} + \cdots + a_{n-k}$, but this is a polynomial in x^t and has no roots of multiplicity more than n/t. □

Proposition 6.3. *Let t be an integer with $t > 1$.*

(i) If n is odd, and j is minimal such that $0 \leq j \leq n$ and $(n-2j, t) = 1$, then among the degrees of any hsop at least $\lfloor (n-j)/t \rfloor$ are divisible by $2t$.

(ii) If n is even, and j is minimal with $0 \leq j \leq \frac{1}{2}n$ and $(\frac{1}{2}n - j, t) = 1$, then among the degrees of any hsop at least $\lfloor (n-j)/t \rfloor$ are divisible by t. □

Corollary 6.4. *Let $t = p^e$ be a power of a prime p, where $e > 0$.*

(i) Suppose $p = 2$. If n is odd, then among the degrees of any hsop at least $\lfloor n/t \rfloor$ are divisible by $2t$. If $n/2$ is odd, then at least $\lfloor n/t \rfloor$ degrees are divisible by t. If $4|n$, then at least $\lfloor (n-2)/t \rfloor$ degrees are divisible by t.

(ii) Suppose $p > 2$. Among the degrees of any hsop at least $\lfloor (n-1)/t \rfloor$ are divisible by t. □

For example, there exist homogeneous systems of parameters with degree sequences 4 ($n = 3$); 2, 3 ($n = 4$); 4, 8, 12 ($n = 5$); 2, 4, 6, 10 ($n = 6$); 4, 8, 12, 12, 20 and 4, 8, 8, 12, 30 ($n = 7$); 2, 3, 4, 5, 6, 7 ($n = 8$).

7 Écritures minimales

Dixmier [6] defines an *écriture minimale* of the Poincaré series as an expression $P(t) = a(t)/\prod(t^{d_i} - 1)$ with minimal $a(1)$ (or, equivalently, with minimal $\prod d_i$; indeed, $\lim_{t\to 1}(t-1)^{n-2}P(t) = a(1)/\prod d_i$). He gives the example of V_7 where $P(t) = a(t)/\prod(t^{d_i} - 1) = b(t)/\prod(t^{e_i} - 1)$ with $d_i = 4, 8, 12, 12, 20$ and $e_i = 4, 8, 8, 12, 30$, and there exist systems of parameters of degrees 4, 8, 12, 12, 20 and of degrees 4, 8, 8, 12, 30.

In our case $n = 9$, in view of the restrictions given in the previous section, the Poincaré series can be written in precisely five minimal ways:

degree $a(t)$	degrees of factors in denominator
66	4, 8, 10, 12, 12, 14, 16
74	4, 4, 10, 12, 14, 16, 24
78	4, 4, 8, 12, 14, 16, 30
86	4, 4, 8, 10, 12, 16, 42
90	4, 4, 8, 10, 12, 14, 48

and we saw that the first corresponds to a system of parameters. In fact all five do, as one can show by following the approach of Dixmier [7].

Proposition 7.1. (Dixmier [7]) *Let G be a reductive group over $\mathbb{C}$, with a rational representation in a vector space R of finite dimension over $\mathbb{C}$. Let $\mathbb{C}[R]$ be the algebra of complex polynomials on R, $\mathbb{C}[R]^G$ the subalgebra of G-invariants, and $\mathbb{C}[R]^G_d$ the subset of homogeneous polynomials of degree d in $\mathbb{C}[R]^G$. Let V be the affine variety such that $\mathbb{C}[V] = \mathbb{C}[R]^G$. Let $\delta = \dim V$. Let $(q_1, \ldots, q_\delta)$ be a sequence of positive integers. Assume that for each subsequence $(j_1, \ldots, j_p)$ of $(q_1, \ldots, q_\delta)$ the subset of points of V where all elements of all $\mathbb{C}[R]^G_j$ with $j \in \{j_1, \ldots, j_p\}$ vanish has codimension not less than p in V. Then $\mathbb{C}[R]^G$ has a system of parameters of degrees $q_1, \ldots, q_\delta$.*

Dixmier gives the covariant $l := (f, f)_8$ and invariants q_j of degree j ($j = 4, 8, 10, 12, 14, 16$) such that if $l = 0$ and all q_j vanish then f belongs to the nullcone. It follows that the set of elements in V where $l = 0$ and p of the invariants q_j vanish has codimension not less than $p + 1$.

Note that when all invariants of degree $3j$ vanish then also all invariants of degree j vanish. Therefore, each of the above five sequences has the property that a subsequence σ of length $p+1$ contains at least p distinct elements, and the set of elements in V where $l = 0$ and all invariants of the degrees in σ vanish has codimension not less than $p+1$.

Let $[j_1, \dots, j_p]'$ be the codimension in V of the set of elements where $l \neq 0$ and all invariants of degrees in $\{j_1, \dots, j_p\}$ vanish. In order to show that each of the five sequences above is the sequence of degrees of a system of parameters it suffices to show that $[4,14]' \geq 3$, $[4,10,14]' \geq 4$, $[4,8,10,14]' \geq 5$, $[4,8,14,16,30]' \geq 6$, $[4,8,10,16,42]' \geq 6$, given that Dixmier already proved the requirements of the proposition for the first sequence.

We did this, using instead of 'all invariants of degree j' the invariants $p_4, q_4, p_8, p_{10}, p_{12}, p_{14}, p_{16}$ defined by Dixmier, and moreover p_{30} and p_{42} found by putting $\tau_1 := (\psi_8, \psi_{10})_0 \in V_{6,10}$, $\tau_2 := (\psi_8, \psi_{10})_1 \in V_{4,10}$, $\tau_3 := (\psi_9, \psi_{10})_0 \in V_{6,14}$, $\tau_4 := (\psi_9, \psi_{10})_1 \in V_{4,14}$, $p_{30} := ((\tau_1, \tau_1)_4, \tau_2)_4$, $p_{42} := ((\tau_3, \tau_3)_4, \tau_4)_4$. The details are very similar to the computation made by Dixmier. The only less trivial part was to show that $[4,10,14]' \geq 4$, which was done using the computer algebra system Singular. Thus:

Proposition 7.2. *The ring of invariants of* V_9 *has systems of parameters with each of the five sequences of degrees* 4, 8, 10, 12, 12, 14, 16 *and* 4, 4, 10, 12, 14, 16, 24 *and* 4, 4, 8, 12, 14, 16, 30 *and* 4, 4, 8, 10, 12, 16, 42 *and* 4, 4, 8, 10, 12, 14, 48. □

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