The unique coclique extension property for apartments of buildings

A. E. Brouwer, J. Draisma & Ç. Güven

16 November 2022

Abstract

We show that the Kneser graph of objects of a fixed type in a building of spherical type has the unique coclique extension property when the corresponding representation has minuscule weight and also when the diagram is simply laced and the representation is adjoint.

1 Introduction

While studying generalizations of the Erdős-Ko-Rado theorem (§2) and the chromatic number of Kneser-type graphs, one needs information about maximal cocliques of near-maximum size, cf. [1, 3–5, 12, 17, 20–22]. In this note we describe a simple construction that in the most interesting cases produces all such near-maximum cocliques.

We say that a pair (Γ, Σ) consisting of a graph and an induced subgraph has the *unique coclique extension property* when every maximal coclique C of Σ is contained in a unique maximal coclique D of Γ. Then D necessarily consists of all vertices x of Γ such that $C \cup \{x\}$ is a coclique, and this property claims that D thus defined is, indeed, a coclique.

As it turns out, Erdős-Ko-Rado type results for set systems and for their q-analogs (systems of subspaces) are very similar. The unique coclique extension property explains part of this similarity, giving a map from cocliques in an apartment of a building to cocliques in the building.

Our Kneser graphs Γ are defined on the objects (flags) of some fixed type J in a building of spherical type, adjacent when they are in opposite chambers. The subgraphs Σ are those induced on such objects in a fixed apartment.

Given a group G with BN-pair (B, N) (cf. [6]) and Weyl group (W, R) and $J \subseteq R$ one has a building with set of chambers G/B and standard apartment WB/B. If the building is spherical then W is finite and has a longest element w_0. Let $J \subseteq R$ and put $X = \langle R \setminus J \rangle$, a subgroup of W. Put $P = BXB$, a parabolic subgroup of G. The vertices of the Kneser graph Γ of type J are the cosets gP with $g \in G$, where gP is adjacent to hP when $Phg^{-1}P = Pw_0P$. The vertices of Σ are the cosets wP with $w \in W$.

Let now G be a reductive linear algebraic group (cf. [7]) defined over the field F. Then the group $G(F)$ has a BN-pair. Algebraic groups give rise to buildings with further structure, and we can use their representation theory. We shall find that the unique coclique extension property holds in cases where the representation has minuscule weight or where the diagram is simply laced and the representation is adjoint.

Theorem 1.1 *Let $X_{n,i}$ be one of $A_{n,i}$ ($1 \leq i \leq n$), $B_{n,1}$, $B_{n,n}$, $C_{n,1}$, $D_{n,1}$, $D_{n,2}$, $D_{n,n}$, $E_{6,1}$, $E_{6,2}$, $E_{7,1}$, $E_{7,7}$, $E_{8,8}$, $G_{2,1}$.*

Let Γ be the Kneser graph on the objects of type i in a building belonging to a Chevalley group with diagram X_n, and let Σ be the subgraph induced on an apartment. Then the pair (Γ, Σ) has the unique coclique extension property. The same holds when Γ is the Kneser graph on the objects of type $\{1, n\}$ in a building with diagram A_n.

The proof of this theorem is spread out over several sections. In §3 we present a simple linear algebra argument that will be used to establish most cases of our theorem, applicable when adjacency in Γ corresponds to the non-vanishing of a certain bilinear map. The cases $A_{n,i}$, $A_{n,\{1,n\}}$, $B_{n,1}/C_{n,1}/D_{n,1}$, $D_{n,2}$ are treated in §§4, 5, 6, 7, respectively, using geometric arguments. Using representation theory of algebraic groups, the cases $B_{n,n}$, $D_{n,n}$, $E_{6,1}$, and $E_{7,7}$ (corresponding to a representation with minuscule weight) follow in §8, while the cases $D_{n,2}$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, and $A_{n,\{1,n\}}$ (where the representation is adjoint) follow in §9.

Some examples of cases where the unique coclique extension property fails are given in §10. Given some examples or nonexamples, §11 discusses further derived (non)examples. For example, the unique coclique extension property holds for type $A_{n,\{1,2\}}$ ($n \geq 2$).

2

2　The Erdős-Ko-Rado theorem

Let n, k be positive integers with $n > 2k$. The Erdős-Ko-Rado theorem [13] affirms that an intersecting family of k-subsets of an n-set has size at most $\binom{n-1}{k-1}$ (and specifies the families attaining this bound). The Hilton-Milner theorem [18] specifies the size (namely $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$) and structure of the second largest maximal intersecting families.

The Kneser graph $K(n, k)$ is the graph on the k-subsets of an n-set, adjacent when disjoint. These two theorems give the size and structure of the largest and next largest maximal cocliques in the Kneser graph.

These theorems have q-analogs. Hsieh [19] showed that a system of non-trivially intersecting k-subspaces of an n-space over $\mathbb{F}_q$ has size at most $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. and in [2] it was shown that the next largest size is $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k$.

This setup can be generalized. Given a spherical building (that is, one with a finite Coxeter group W), the Kneser graph Γ of type J has as vertices the objects of type J, adjacent when they are in opposite chambers.[1] Parameter information for these graphs was given in [9, 11].

The Kneser graph $K(n, k)$ and its q-analog are the special cases of the Kneser graph of type k in an apartment or building of type A_{n-1}.

For analogs for maximal singular subspaces in a polar space of rank n, see [25], [23]. Tanaka [26] classifies the subsets in certain association schemes of which width plus dual width equals the number of classes, and derives EKR-type results. For a book-length discussion of EKR-type results, see [15].

In the cases where the unique coclique extension property holds it often yields the largest cocliques in the Kneser graph. For example, for $A_{n,\{1,n\}}$ (over $\mathbb{F}_q$) the maximum size is $1 + 2q + 3q^2 + \cdots + nq^{n-1}$ and examples of this size arise using this construction (see §5).

3　The coclique extension lemma

Let V, V' be vector spaces and $\mu : V \times V \to V'$ a bilinear map that is reflexive (i.e., satisfies $\mu(x, y) = 0$ if and only if $\mu(y, x) = 0$). For $A \subseteq V$, write $A^\perp = \{v \in V \mid \mu(v, a) = 0 \text{ for all } a \in A\}$.

---

[1]Two singular k-subspaces A, B of a polar space of rank n are adjacent in the Kneser graph when $A^\perp \cap B = 0$, a relation that is stricter than the disjointness $A \cap B = 0$.

Suppose the vertex set $V\Gamma$ of a graph Γ can be embedded into the projective space PV corresponding to V in such a way that vertices x, y are nonadjacent in Γ if and only if $\mu(x, y) = 0$ (and in particular $\mu(x, x) = 0$ for all vertices).

Lemma 3.1 *Let C be a coclique in Γ, so that $C \subseteq C^\perp$, and let D be the set of vertices x of Γ for which $C \cup \{x\}$ is also a coclique, so that $D = C^\perp \cap V\Gamma$. If D is contained in $\langle C \rangle$, the linear span of C in V, then D is a coclique.* $\square$

This will be our main tool in many of the cases.

4   Subspaces of a projective space

Fix integers m, n, where $1 \le m \le \frac{1}{2}n$. Let $N = \{1, \ldots, n\}$. Let F be a division ring, let V be an n-dimensional left vector space over F with basis $\{e_1, \ldots, e_n\}$, and let PV be the corresponding projective space. Each m-set $K \subseteq N$ determines an m-space $\phi K \le PV$ by $\phi K = \langle e_i \mid i \in K \rangle$.

Proposition 4.1 *Let C be a maximal collection of pairwise intersecting m-subsets of N. Let D be the collection of all m-spaces in PV that meet ϕK for all $K \in C$. Then D is a maximal collection of pairwise intersecting m-subspaces of PV.*

(Here and elsewhere, dimensions are vector space dimensions.)

First proof (for the case of a field F). Let $\bigwedge V$ be the exterior algebra of V. Map m-subspaces of PV to projective points in $P(\bigwedge V)$ via

$$\psi : U = \langle u_1, \ldots, u_m \rangle \mapsto \langle u_1 \wedge \ldots \wedge u_m \rangle.$$

Now $U \cap U' \ne 0$ if and only if $\psi U \wedge \psi U' = 0$. For $M = \{i_1, \ldots, i_m\} \subseteq N$ with $i_1 < \ldots < i_m$ let $e_M = e_{i_1} \wedge \ldots \wedge e_{i_m}$. The e_M form a basis for the degree m part of $\bigwedge V$. Suppose $U \in D$, where $\psi U = \langle \sum \alpha_M e_M \rangle$. Then $\alpha_M = 0$ whenever M is disjoint from some $K \in C$. Indeed, if $K \cap M = \emptyset$, then the coefficient of $e_{K \cup M}$ in $\psi U \wedge \psi \phi K$ is α_M up to a nonzero constant.

So, if $\alpha_M \ne 0$ then M meets all $K \in C$, and, since C is maximal, $M \in C$. We see that the 1-space ψU is contained in $\langle \psi \phi C \rangle$. If $U, U' \in D$, then $\psi U, \psi U' \subseteq \langle \psi \phi C \rangle$ implies $\psi U \wedge \psi U' = 0$ so that $U \cap U' \ne 0$. Maximality of D is clear. $\square$

Note that the proof above implements the argument from §3. If F is not commutative, we need a different argument.

Second proof (for the general case). Given a matrix A with entries in F, let its *rank* be the dimension of the left vector space spanned by its rows, or, equivalently, the dimension of the right vector space spanned by its columns.

Given a matrix A with entries in F and columns indexed by N, we find a matroid M_A on N by letting the rank of $M \subseteq N$ be the rank of the submatrix of A on the columns indexed by M.

Since V has a fixed basis $(e_i)_i$, we can regard each $v \in V$ as a row vector. Accordingly, we regard V as a left vector space over F. Given a (left F-) subspace U of V, let $A(U)$ be a matrix of which the rows form a basis of U. We find a matroid M_U on N by taking $M_U = M_{A(U)}$. It is independent of the choice of $A(U)$.

An m-space U meets ϕK if and only if no basis of M_U is disjoint from K. Hence $U \in D$ if and only if each basis of M_U meets all elements of C, i.e., by maximality of C, if and only if each basis of M_U belongs to C. Now the claim follows from:

Lemma. *Let U, U' be disjoint subspaces of V. Then $M_U, M_{U'}$ have disjoint bases.*

Proof. Let M be the union matroid of $M_U, M_{U'}$, that is the matroid of which the independent sets are the unions of an independent set of M_U and one of $M_{U'}$. As is well known (see, e.g., [24], Ch. 42), the rank function of M is given by

$$r(K) = \min_{L \subseteq K} \left(|K \setminus L| + r_U(L) + r_{U'}(L) \right),$$

where $r_U, r_{U'}$ are the rank functions of $M_U, M_{U'}$. Let $\dim U = m$, $\dim U' = m'$. We have to show that $r(N) = m + m'$, that is, that

$$n - |L| + r_U(L) + r_{U'}(L) \geq m + m'$$

for all $L \subseteq N$. But $r_U(L) + r_{U'}(L) \geq r_{U+U'}(L)$ and $m + m' - r_{U+U'}(L) \leq n - |L|$ since the subspace of $U + U'$ consisting of the vectors vanishing on the positions in L is contained in the subspace of V of such vectors. □□

5 Point-hyperplane flags in a projective space

In this section we study the Kneser graphs for type $A_{n,\{1,n\}}$.

Let V be an $(n+1)$-dimensional left vector space over the division ring F, and let Γ be the graph of which the vertices are the point-hyperplane flags (P, H) (where points are 1-spaces), where $(P, H) \sim (Q, I)$ when $P \not\subseteq I$ and $Q \not\subseteq H$.

Let V have basis $\{e_i \mid i \in N\}$, where $N = \{0, \dots, n\}$. Let Σ be the subgraph of Γ induced on the set of $n(n + 1)$ vertices $\{(i, N \setminus j) \mid i \neq j\}$, where $(i, N \setminus j)$ abbreviates $(\langle e_i \rangle, \langle \{e_h \mid h \neq j\} \rangle)$. A maximal coclique C in Σ has size $\binom{n+1}{2}$ and contains precisely one vertex from every pair $(i, N \setminus j)$, $(j, N \setminus i)$.

Fix C, and let D be the set of vertices of Γ not adjacent to any vertex in C. We claim that D is a coclique in Γ. Indeed, suppose that D contains adjacent vertices $(\langle p \rangle, I)$ and $(\langle q \rangle, J)$. Let $p = \sum \alpha_i e_i$ and $q = \sum \beta_j e_j$. Since $p \notin J$ there is an i with $\alpha_i \neq 0$ and $e_i \notin J$. Similarly, there is a j with $\beta_j \neq 0$ and $e_j \notin I$. Since $p \in I$ we have $i \neq j$. W.l.o.g., let $(i, N \setminus j) \in C$. Both $e_i \in J$ and $q \in \langle N \setminus j \rangle$ are impossible, contradiction. It follows that the unique coclique extension property holds here.

Now let $F = \mathbb{F}_q$. It was shown in [3] that the largest cocliques in Γ have size $1 + 2q + 3q^2 + \cdots + nq^{n-1}$, that if we pick C so that it contains $(i, N \setminus j)$ precisely when $i < j$, then D will be a maximum size coclique in Γ, and that all maximum cocliques in Γ are obtained in this way (for a suitable choice of the apartment Σ and ordering of N).

6   Points in a polar space

A *polar space of rank n* is a building of type C_n with points and lines being the objects of types 1 and 2. All objects are singular subspaces. The Kneser graph Γ on the points is the noncollinearity graph. An apartment in this building is a cross polytope on $2n$ vertices (for $n = 2$ a square, for $n = 3$ an octahedron), and the Kneser graph Σ is nK_2, the disjoint union of n copies of K_2.

Every maximal coclique C of Σ has size n, and is contained in a unique maximal singular subspace D, that is, in a unique maximal coclique of Γ. It follows that the unique coclique extension property holds.

The buildings of Chevalley or twisted Chevalley groups of types B_n, C_n, D_n, ${}^2A_{2n-1}$, ${}^2A_{2n}$, and ${}^2D_{n+1}$ are polar spaces of rank n.

For the classical generalized hexagon of type G_2, the points are the points of the polar space of type B_3, and the Kneser graphs of both coincide, and

so do the subgraphs induced on an apartment. It follows that the unique coclique extension property also holds for type $G_{2,1}$.

7 Totally singular lines in an orthogonal space

In this section we study the Kneser graphs for type $D_{n,2}$.

Fix an integer $n \geq 2$. Let $N = \{1, 1', 2, 2', \ldots, n, n'\}$. It is provided with an involution $'$ that interchanges i and i'. Let V be a $2n$-dimensional vector space (over some field F) with basis $\{e_s \mid s \in N\}$ and let Q be the quadratic form on V defined by $Q(x) = x_1 x_{1'} + \ldots + x_n x_{n'}$. Now (V, Q) is a nondegenerate orthogonal space with maximal Witt index. Each pair $\{s, t\} \subseteq N$ determines a line in PV by $\phi\{s, t\} = \langle e_s, e_t \rangle$. This line will be totally singular when $t \neq s'$.

Proposition 7.1 *Let C be a maximal collection of pairs $\{s, t\}$ in N such that $t \neq s'$ and if $\{s, t\} \in C$ then $\{s', t'\} \notin C$. Let D be the collection of all totally singular lines in PV that meet $(\phi K)^\perp$ for all $K \in C$. Then $L^\perp \cap M \neq 0$ for any two lines $L, M \in D$.*

Note that $L^\perp \cap M \neq 0$ if and only if $L \cap M^\perp \neq 0$.

Proof. As before, let ψ map subspaces of PV to points in $P(\bigwedge V)$. For two lines L, M we have $L \cap M^\perp \neq 0$ whenever $\psi L \wedge \psi(M^\perp) = 0$. For $K \in C$ we have $\psi((\phi K)^\perp) = e_{N \setminus K'}$. It follows that the line L with $\psi L = \sum \alpha_P e_P$ (with P running over the 2-subsets of N) intersects $(\phi K)^\perp$ for $K \in C$ if and only if $\alpha_{K'} = 0$. Let $L \in D$, and suppose ψL involves $e_{i,i'}$ for some i. Since L is totally singular, it follows that ψL also involves $e_{j,j'}$ for some $j \neq i$. The collection C contains precisely one of $\{s, t\}$ and $\{s', t'\}$ whenever $t \neq s'$, so we may assume that C contains $\{i, j\}$ and $\{i, j'\}$, so that $\alpha_{\{i', j\}} = \alpha_{\{i', j'\}} = 0$. Let $L = \langle u, v \rangle$ with $u = \sum u_s e_s$ and $v = \sum v_s e_s$ where $u_{i'} = 0$. Then $v_{i'} \neq 0$ and $u_j = u_{j'} = 0$ so that ψL does not involve $e_{j,j'}$, a contradiction. So, $\alpha_P = 0$ when $P = \{i, i'\}$. We proved that $\psi L \in \langle \psi \phi C \rangle$. Now the proof finishes as the first proof of Proposition 4.1. $\qquad \square$

The above proved the unique coclique extension property for the Kneser graph of type $D_{n,2}$. It also holds for the disjointness graph in this geometry.

Proposition 7.2 *Let C be a maximal collection of pairwise intersecting 2-subsets $\{s, t\}$ of N with $t \neq s'$. Let D be the collection of all totally singular*

lines in PV that meet ϕK for all $K \in C$. Then D is a maximal collection of pairwise intersecting totally singular lines in PV.

Proof. This is trivial: one finds either all lines in a totally singular plane or all totally singular lines on a fixed point. □

8 Minuscule weights

Let G be a split connected reductive algebraic group defined over a field F, T a maximal split torus, and V an irreducible algebraic representation of G over F. For each character χ of T, let V_χ be the subspace of V on which T acts with that character. Assume that V is minuscule, i.e., that W acts transitively on the weights χ for which V_χ is nonzero. Then it follows from irreducibility that all weight spaces are 1-dimensional. Fix such a weight λ, let $v_\lambda \in V_\lambda(F)$ be nonzero, and let P be the stabilizer of v_λ.

Now the Kneser graph Γ is the graph with vertex set $G(F)/P(F)$, embedded in the projective space corresponding to V as the orbit of $\langle v_\lambda \rangle$ under $G(F)$, and $V\Sigma$ is the orbit of $\langle v_\lambda \rangle$ under the Weyl group W. The neighbours of a vertex in $V\Sigma$, corresponding to the weight $\lambda' \in W$, are the vertices corresponding to weights that are at maximal distance from λ'.

We will look for reflexive bilinear maps as in the following definition.

Definition 8.1 *A reflexive bilinear map $V \times V \to V'$, where V, V' are G-modules, is called* good *if it has the following properties: two vertices x, y of Γ are adjacent in Γ if and only if $\mu(x, y) \neq 0$; and, moreover, for each $x \in V\Sigma$, the nonzero vectors $\mu(x, y)$ with $y \in V\Sigma$ are linearly independent.*

Proposition 8.2 *(i) For each minuscule representation V, there exists a good bilinear map $\mu : V \times V \to V'$.*

(ii) The pair (Γ, Σ) has the unique coclique extension property.

This settles the cases $A_{n,j}$ ($1 \leq j \leq n$), $B_{n,n}$, $C_{n,1}$, $D_{n,1}$, $D_{n,n}$, $E_{6,1}$, $E_{7,7}$.

Proof. Let us do part (ii) first, given part (i).

Assume that we have a good bilinear map $\mu : V \times V \to V'$. The module V is a direct sum of 1-dimensional weight spaces permuted by W and spanned by the vertices of Σ, so that $\dim V = |V\Sigma|$. Pick a basis vector e_s for each $s \in \Sigma$. For $x \in V\Gamma$ write $x = \sum x_s e_s$. If $x \in C^\perp$, then $0 = \mu(x, e_c) =$

$\sum x_s \mu(e_s, e_c)$ for each $c \in C$. Since, by assumption, the nonzero vectors $\mu(e_s, e_c)$ are linearly independent, we find that for all s with $x_s \neq 0$, we have $\mu(e_s, e_c) = 0$ for all c. Since C is maximal, any s with this property lies in C, and we showed that $x \in \langle C \rangle$.

It remains to find a good bilinear map $\mu : V \times V \to V'$.

We already did $A_{n,j}$ (using the bilinear map $\mu : V \times V \to \bigwedge V_0$, where $V = \bigwedge^j V_0$ and V_0 is the natural module for SL_{n+1} of dimension $n + 1$), and $C_{n,1}$ and $D_{n,1}$ (using the natural symplectic or symmetric bilinear form $\mu : V \times V \to F$).

More generally, if V is a self-dual representation, then there is a G-equivariant reflexive bilinear form $\mu : V \times V \to F$. In that case, every vertex in $V\Sigma$ has a unique neighbour in $V\Sigma$, namely, the one with the opposite weight, so the condition on linear independence is automatically satisfied. That settles $B_{n,n}$, $D_{n,n}$ (n even), and $E_{7,7}$, in addition to $C_{n,1}$ and $D_{n,1}$.

For $D_{n,n}$ (n odd), Γ is the graph on maximal isotropic subspaces in a $2n$-dimensional orthogonal space V_0 that intersect one given such space in an odd-dimensional subspace. Two are adjacent if they intersect in a 1-dimensional space. Here V is the spin representation of the spin group G. The spin representation has a nonzero reflexive G-equivariant bilinear map $\mu : V \times V \to V_0$ (symmetric when n is 1 modulo 4, skew-symmetric when n is 3 modulo 4).[2] If $a, b \in V$ represent isotropic n-dimensional spaces $A, B \subseteq V_0$ that intersect in an odd-dimensional space C, and if $\mu(a, b)$ is nonzero, then equivariance implies that C is one-dimensional and spanned by $\mu(a, b)$. Furthermore, if $e_1, e_{1'}, \ldots, e_n, e_{n'}$ is the basis of V_0 discussed in §7, and $a \in V$ is the vector representing $\langle e_1, \ldots, e_n \rangle$, then the 2^{n-1} basis vectors of the spin representation each represent a space spanned by e_i for an odd number of $i \in \{1, \ldots, n\}$, and $e_{i'}$ for the remaining i. It then follows that

---

[2]If the characteristic is not 2, μ can be taken to be the map β_1 from Chevalley [10], III.4.4, and a similar construction works in characteristic 2. Much more generally, the existence of μ follows from [8, Theorem 4.3.2], as pointed out to us by Shrawan Kumar. Our V is the Weyl module $V(\omega_n)$, where the n-th fundamental weight ω_n is the minuscule weight under consideration, and the theorem there says that there is a unique surjective equivariant bilinear map from the submodule M of $V \otimes V$ generated by the tensor product $v_{\omega_n} \otimes v_{w_0 \omega_n}$ into $V(\overline{\omega_n + w_0 \omega_n})$. Here w_0 is the longest element in the Weyl group, v_λ spans the (one-dimensional) weight space in V corresponding to λ, and $\overline{\lambda}$ is the unique dominant weight in the Weyl group orbit of λ. In the current case, a straightforward check shows that M is all of $V \otimes V$, $w_0 \omega_n = -\omega_{n-1}$ (for D_n with n odd, w_0 is the order-two automorphism of the diagram followed by -1), and $\overline{\omega_n - \omega_{n-1}} = \omega_1$, the highest weight of the standard representation.

precisely n of these intersect $\langle e_1, \dots, e_n \rangle$ in a 1-dimensional space, and as b ranges over the corresponding basis vectors, $\mu(a, b)$ ranges over $e_1, \dots, e_n$. So these are linearly independent as desired.

For $E_{6,1}$, the module V is 27-dimensional, and the dual module V' is that of $E_{6,6}$. There is a nonzero symmetric G-equivariant bilinear map $V \times V \to V'$.[3] It has the property that $\mu(x, y) = z$, where z is the unique symplecton containing x, y when x, y are noncollinear, and $z = 0$ otherwise. The apartment is the Schläfli graph (27 vertices, valency 16). For fixed c there are 10 nonadjacent vertices, and the 10 symplecta $\mu(c, s)$ are distinct and hence linearly independent. $\qquad\square$

9 Adjoint representation

Assume that the algebraic group G has simply laced Dynkin diagram, and consider its adjoint representation V, the Lie algebra of G. In this case, Γ is the orbit of root vectors, and two are adjacent if they generate a Lie subalgebra isomorphic to $\mathfrak{sl}_2(F)$, i.e., with respect to a suitable choice of maximal torus, they correspond to opposite roots.

Proposition 9.1 *There is a good bilinear form $V \times V \to F$, and the unique coclique extension property holds.*

This covers $D_{n,2}$ $(n \geq 4)$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, and $A_{n,\{1,n\}}$ $(n \geq 2)$.

Proof. The module $V = V(\lambda)$ carries the structure of a Lie algebra $\mathfrak{g}$. Let Φ be the root system. Then $\dim V = |\Phi| + \ell$ where ℓ is the Lie rank of G, the dimension of a Cartan subalgebra $\mathfrak{h}$. Pick a Chevalley basis consisting of e_s for $s \in \Phi$ and e_i for $1 \leq i \leq \ell$, where $V\Sigma$ is the collection of root spaces $\langle e_s \rangle$ for $s \in \Phi$, and $\mathfrak{h}$ is spanned by the e_i.

Let $r \in \Phi$, and consider $[e_r, [e_r, v]]$ for arbitrary $v \in V$. If $v = e_s$ for some $s \in \Phi$, then r, s span a 2-dimensional root system, and we see that this is zero, unless $r + s = 0$, in which case it is $-2e_r$. If $v = h \in \mathfrak{h}$, then $[e_r, [e_r, v]] = 0$. More generally, for $v = h + \sum v_s e_s$, we find $[e_r, [e_r, v]] = -2v_{-r}e_r$. Let us call an element x of a Lie algebra *extremal* when $[x, [x, y]]$ is a multiple of x for all y. (The usual definition adds a requirement for the case of characteristic

---
[3]Also this is a special case of [8, Theorem 4.3.2].

2 that we don't need.) We just observed that e_r is extremal. It follows that x is extremal for every $x \in V\Gamma$.

The graph Σ has valency 1, so that a maximal clique C contains exactly one from each pair of opposite root vectors.

The usual Killing form has a constant factor that makes it identically zero in small characteristics. Let μ be the reduced Killing form (cf. [14], §8 and [16], §5), then $\mu(e_r, e_{-r}) = 1$, so that the bilinear form μ is good.

We want to show that if $x, y \in C^\perp \cap V\Gamma$, then $\mu(x, y) = 0$. Let $x = x_0 + \sum x_s e_s$ and $y = y_0 + \sum y_s e_s$ with $x_0, y_0 \in \mathfrak{h}$. If $\langle e_c \rangle \in C$, then $x_{-c} = \mu(x, e_c) = 0$, and similarly $y_{-c} = 0$. It follows that $\mu(x, y) = \mu(x_0, y_0)$. If x_0 lies in the radical of μ, then $\mu(x_0, y_0) = 0$. If not, then there is a root $t \in \Phi$ with $t(x_0) \neq 0$. If necessary replace t by $-t$ to make sure that $\langle e_t \rangle \notin C$. Since x is extremal, $[x, [x, e_t]]$ is a multiple of x. On the other hand, the coefficient of e_t in this expression is $t(x_0)^2 \neq 0$ (since $x_s x_{-s} = 0$ for all $s \in \Phi$). This is a contradiction. $\qquad\square$

10 Nonexamples

The unique extension property does not hold for arbitrary types. We give counterexamples for types $A_{n-1,\{i,n-i\}}$ $(1 < i < n/2)$, $B_{3,2}$, $C_{3,3}$, $D_{4,\{3,4\}}$, and $D_{5,3}$.

Proposition 10.1 *The unique coclique extension property does not hold for the Kneser graph on the flags of type $\{i, n-i\}$ in a building with diagram A_{n-1}, where $1 < i < n/2$.*

Proof. Let V be a vector space of dimension n with basis $e_1, \ldots, e_n$. Put $u = e_1 + e_2$, $v = e_1 + e_n$. Let $A = \langle u, e_3, \ldots, e_{i+1} \rangle$ and $A' = \langle v, e_{n-1}, \ldots, e_{n-i+1} \rangle$ so that A and A' are i-spaces. Let $B = \langle u, e_3, \ldots, e_{n-i}, e_n \rangle$ and $B' = \langle v, e_{n-1}, \ldots, e_{i+2}, e_2 \rangle$ so that B and B' are $(n-i)$-spaces. Now the flags $F = (A, B)$ and $F' = (A', B')$ are adjacent since $A \cap B' = A' \cap B = 0$. Note that F is mapped to F' by the coordinate permutation $(2, n)(3, n-1)\ldots$.

The graph Σ has valency 1 and each choice of one vertex from each edge of Σ yields a maximal coclique C. We can find a maximal coclique C compatible with F, F' when in no edge of Σ both endpoints are adjacent to either F or F'. Let $N = \{1, \ldots, n\}$. The vertices of Σ are pairs (S_I, S_J) where

11

$S_I = \langle e_i \mid i \in I \rangle$ and $|I| = i$, $|J| = n - i$, $I \subseteq J$. The unique neighbour of (S_I, S_J) is $(S_{N \setminus J}, S_{N \setminus I})$. If (S_I, S_J) is adjacent to F, then $S_I \cap B = S_J \cap A = 0$, so that $I = \{j, n-i+1, \ldots, n-1\}$ and $J = \{j, i+2, \ldots, n\}$, where $j \in \{1, 2\}$. If (S_I, S_J) is adjacent to F', then $I = \{k, i+1, \ldots, 3\}$ and $J = \{k, n-i, \ldots, 2\}$ where $k \in \{1, n\}$. Altogether 4 vertices in Σ are adjacent to either F or F', and for $i > 1$ this set of 4 does not contain any edge. $\qquad\square$

$B_{3,2}$ nonexample: Let char $F \neq 2$ and let $V_0 = F^7$, with basis vectors e_i, $1 \leq i \leq 7$. Provide V_0 with the nondegenerate quadratic form $Q(x) = x_1 x_2 + x_3 x_4 + x_5 x_6 - x_7^2$. The geometry of totally singular subspaces of (V_0, Q) has type B_3. For type B_3, the adjoint representation is $\wedge^2 V_0$, corresponding to $B_{3,2}$, the geometry of totally singular lines. Consider the apartment with points $\langle e_i \rangle$, $1 \leq i \leq 6$, and let C consist of the six totally singular lines $13, 14, 25, 26, 35, 36$, where ij abbreviates $\langle e_i, e_j \rangle$. Pick $u = e_1$, $v = e_3 + e_4 + e_7$ and $u' = e_2$, $v' = e_5 + e_6 + e_7$. Then $\langle u, v \rangle$ and $\langle u', v' \rangle$ are totally singular and adjacent in the Kneser graph and both nonadjacent to all vertices of C.

$C_{3,3}$ nonexample: Let char $F \neq 2$ and let $V_0 = F^6$, with basis vectors e_i, $1 \leq i \leq 6$. Provide V_0 with the nondegenerate symplectic form $f(x, y) = x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + x_5 y_6 - x_6 y_5$. The geometry of totally isotropic subspaces of (V_0, f) has type C_3. Consider the apartment with points $\langle e_i \rangle$, $1 \leq i \leq 6$, and let C consist of the four totally isotropic planes $135, 146, 236, 245$ where ijk abbreviates $\langle e_i, e_j, e_k \rangle$. The two planes $\langle e_1, e_3 + e_6, e_4 + e_5 \rangle$ and $\langle e_1 + e_3 - e_5, e_2 + e_6, e_4 + e_6 \rangle$ are totally isotropic and disjoint (since char $F \neq 2$), i.e., adjacent in the Kneser graph, while nonadjacent to all vertices of C.

Planes in D_4 nonexample: Let $N = \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and let $'$ be the involution on N that maps i to i'. Let char $F = 2$ and let $V_0 = F^8$, with basis vectors e_i $(i \in N)$ and provided with the nondegenerate quadratic form $Q(x) = x_1 x_{1'} + x_2 x_{2'} + x_3 x_{3'} + x_4 x_{4'}$. The geometry of totally singular subspaces of (V_0, Q) has type D_4. Consider the apartment with points $\langle e_i \rangle$, $i \in N$. Let $\pi = \langle e_1 + e_2, e_{1'} + e_{2'}, e_4 \rangle$ and $\pi' = \langle e_{1'} + e_3, e_1 + e_{3'}, e_{4'} \rangle$. Then π, π' are totally singular planes, adjacent in the Kneser graph on the planes in D_4 (since $\pi^\perp \cap \pi' = 0$). Here Σ has valency 1, and one checks that there is no edge in Σ such that both ends are adjacent to π or π', so there is a maximal coclique C compatible with both π and π', and the unique coclique extension property fails. This was the case $D_{4,\{3,4\}}$. Of course this also means that it fails for $D_{n,3}$ for all $n \geq 5$.

11 Varying J

Given a building of spherical type, let Γ, Γ' be the Kneser graphs on the objects of type J, J', respectively, where $J \subseteq J'$. Let (W, R) be the Coxeter system of the building, and put $X = \langle R \setminus J \rangle$, $X' = \langle R \setminus J' \rangle$, and $P = BXB$, $P' = BX'B$, so that the vertices of Γ, Γ' can be viewed as left cosets of P, P', respectively.

Lemma 11.1 *The canonical map $\phi : \Gamma' \to \Gamma$ sending gP' to $gP'P = gP$ is a homomorphism, that is, sends edges to edges.*

Proof. Since $J \subseteq J'$, we have $X' \leq X$ and $P' \leq P$. If $gP' \sim hP'$ in Γ', that is, if $Bw_0B \subseteq P'g^{-1}hP'$, then also $Bw_0B \subseteq Pg^{-1}hP$, that is, $gP \sim hP$ in Γ. $\square$

Lemma 11.2 *Suppose $J^{w_0} = J$. If a' is a vertex of Γ' and a, b are adjacent vertices of Γ, where $\phi(a') = a$, then there is a vertex b' of Γ', adjacent to a', with $\phi(b') = b$. That is, edges can be lifted.*

Proof. Let $a' = yP'$, $a = yP$, $b = zP$. Since $J^{w_0} = J$, we have $w_0X = Xw_0$. Since yP, zP are adjacent, we have $Bw_0B \subseteq Pz^{-1}yP$, so that $z^{-1}y \in Pw_0P = BXw_0XB = BXw_0B \subseteq Pw_0B$, and $w_0B = p^{-1}z^{-1}yB$ for some $p \in P$. It follows that the vertex $a' = yP'$ is adjacent to $b' = zpP'$. $\square$

If the unique coclique extension property fails for objects of some type J in a building of spherical type with Coxeter system (W, R), where $J^{w_0} = J$, and $J \subseteq J' \subseteq R$, then it also fails for objects of type J'.

Proposition 11.3 *Let Γ, Γ' be the Kneser graphs on the objects of type J, J', respectively, in a building of spherical type. Let Σ, Σ' be the subgraphs of Γ, Γ', respectively, induced on an apartment. If $J^{w_0} = J$, and $J \subseteq J'$, and the pair (Γ', Σ') has the unique coclique extension property, then also (Γ, Σ) has this property.*

Proof. We can take Σ, Σ' to be the subgraphs induced on WP/P and WP'/P'.

Let C be a maximal coclique in Σ. Then $\phi^{-1}C$ is a coclique in Σ'. If $wP \notin C$, there is a $vP \in C$ adjacent to wP, and we can lift this edge and find a neighbour of wP' in $\phi^{-1}C$. Hence $\phi^{-1}C$ is a maximal coclique in Σ'.

If y, z are two vertices of Γ such that both $C \cup \{y\}$ and $C \cup \{z\}$ are cocliques, then both $\phi^{-1}C \cup \phi^{-1}(y)$ and $\phi^{-1}C \cup \phi^{-1}(z)$ are cocliques in Γ', and by the unique coclique extension property of (Γ', Σ'), no vertex of $\phi^{-1}(y)$ is adjacent to a vertex of $\phi^{-1}(z)$. But then also y and z are nonadjacent. $\square$

If the shortest element in $X w_0 X$ equals the shortest element in $X' w_0 X'$, then Γ' is a coclique extension of Γ, and (Γ', Σ') has the unique coclique extension property if and only if (Γ, Σ) has this property. For example, if we label the nodes of the A_n diagram by $1, \ldots, n$, and $J \subseteq \{1, \ldots, j\}$ with $\max J = j \leq (n+1)/2$, then $A_{n,J}$ is a coclique extension of $A_{n,j}$ and so has the unique coclique extension property.

References

[1] A. Blokhuis & A. E. Brouwer, *Cocliques in the Kneser graph on line-plane flags in* $\mathrm{PG}(4, q)$, Combinatorica **37** (2017) 795–804.

[2] A. Blokhuis, A. E. Brouwer, A. Chowdhury, P. Frankl, T. Mussche, B. Patkós & T. Szőnyi, *A Hilton-Milner theorem for vector spaces*, Electr. J. Combin. **17** (2010) R71.

[3] A. Blokhuis, A. E. Brouwer & Ç. Güven, *Cocliques in the Kneser graph on the point-hyperplane flags of a projective space*, Combinatorica **34** (2014) 1–10.

[4] A. Blokhuis, A. E. Brouwer & T. Szőnyi, *On the chromatic number of q-Kneser graphs*, Des. Codes Cryptogr. **65** (2012) 187–197.

[5] A. Blokhuis, A. E. Brouwer & T. Szőnyi, *Maximal cocliques in the Kneser graph on point-plane flags in* $\mathrm{PG}(4, q)$, Europ. J. Comb. **35** (2014) 95–104.

[6] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Springer, 2002.

[7] A. Borel, *Linear algebraic groups*, Springer, 1991.

[8] M. Brion & S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory*, Birkhäuser, 2005.

[9] A. E. Brouwer, *The eigenvalues of oppositeness graphs in buildings of spherical type*, pp. 1–10 in: Combinatorics and graphs, Contemp. Math. **531**, Amer. Math. Soc., Providence, RI, 2010.

[10] C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, 1954. Reprinted in: *The algebraic theory of spinors and Clifford algebras*, Collected Works Vol. 2, Springer, 1997.

[11] J. De Beule, S. Mattheus & K. Metsch, *An algebraic approach to Erdős-Ko-Rado sets of flags in spherical buildings*, `arXiv:2007.01104`.

[12] J. D'haeseleer, K. Metsch & D. Werner, *On the chromatic number of two generalized Kneser graphs*, Europ. J. Comb. **101** (2022) 103474.

[13] P. Erdős, C. Ko & R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961) 313–320.

[14] S. Garibaldi, R. M. Guralnick & D. K. Nakano, *Globally irreducible Weyl modules*, J. Algebra **477** (2017) 69–87.

[15] C. Godsil & K. Meagher, *Erdős-Ko-Rado theorems: Algebraic approaches*, Cambridge Univ. Press, Cambridge, 2015.

[16] B. H. Gross & G. Nebe, *Globally maximal arithmetic groups*, J. Algebra **272** (2004) 625–642.

[17] Ç. Güven, *Buildings and Kneser graphs*, Ph. D. Thesis, Technische Univ. Eindhoven, Eindhoven, 2012.

[18] A. J. W. Hilton & E. C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **18** (1967) 369–384.

[19] W. N. Hsieh, *Intersection theorems for systems of finite vector spaces*, Discrete Math. **12** (1975) 1–16.

[20] K. Metsch, *An Erdős-Ko-Rado result for sets of pairwise non-opposite lines in finite classical polar spaces*, Forum Math. **31** (2019) 491–502.

[21] K. Metsch, *An Erdős-Ko-Rado result for finite buildings of type F_4*, Israel J. Math. **230** (2019) 813–830.

[22] K. Metsch & D. Werner, *Maximal cocliques in the Kneser graph on plane-solid flags in* PG$(6, q)$, Innov. Incidence Geom. **18** (2020) 39–55.

[23] V. Pepe, L. Storme & F. Vanhove, *Theorems of Erdős-Ko-Rado type in polar spaces*, J. Comb. Th. (A) **118** (2011) 1291–1312.

[24] A. Schrijver, *Combinatorial Optimization*, Springer, 2003.

[25] D. Stanton, *Some Erdős-Ko-Rado theorems for Chevalley groups*, SIAM J. Alg. Disc. Math. **1** (1980) 160–163.

[26] H. Tanaka, *Classification of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs*, J. Combin. Th. (A) **113** (2006) 903–910.

aeb@cwi.nl, jan.draisma@unibe.ch, C.Guven@tilburguniversity.edu