Parameters of an association scheme

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In BCN [1], Theorem 12.1.1 the existence of a certain association scheme is claimed, and details are given for \( n = 3 \). As Frédéric Vanhove [2] observed, things are slightly different for odd \( n \geq 5 \). Let us redo his computations.

Let \( q \) be a power of 2, and \( n \geq 3 \). Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{F}_q \) provided with a nondegenerate quadratic form \( Q \). Let \( B \) be the associated symmetric bilinear form, given by \( B(x, y) = Q(x + y) - Q(x) - Q(y) \).

If \( n \) is odd, there will be a nucleus \( N = V^\perp \).

We construct an association scheme with point set \( X \), where \( X \) is the set of projective points not on the quadric \( Q \) and (for odd \( n \)) distinct from \( N \). For \( n = 3 \) and for even \( n \), the relations will be \( R_0, R_1, R_2, R_3 \) where

\[
R_0 = \{(x, x) \mid x \in X\}, \text{ the identity relation;}
R_1 = \{(x, y) \mid x + y \text{ is a hyperbolic line (secant)}\};
R_2 = \{(x, y) \mid x + y \text{ is an elliptic line (exterior line)}\};
R_3 = \{(x, y) \mid x + y \text{ is a tangent}\}.
\]

For odd \( n, n \geq 5 \), it is necessary to distinguish \( R_{3a} \) and \( R_{3n} \), defined by

\[
R_{3a} = \{(x, y) \mid x + y \text{ is a tangent not on } N\};
R_{3n} = \{(x, y) \mid x + y \text{ is a tangent on } N\}.
\]

Note that every line on \( N \) is a tangent, and that for \( n = 3 \) there are no other tangents, so that \( R_{3a} \) is empty. For \( q = 2 \) a hyperbolic line contains only one nonisotropic point, so that \( R_1 \) is empty.

1 Quadric size

The number of \( N \) isotropic projective points on a nonisotropic quadric in \( V \), where \( V \) has vector space dimension \( n \) equals

\[
N = \begin{cases} 
(q^{2m} - 1)/(q - 1) & \text{if } n = 2m + 1 \\
(q^m - \varepsilon)(q^{m-1} + \varepsilon)/(q - 1) & \text{if } n = 2m.
\end{cases}
\]

Equivalently,

\[
N = (q^{n-1} - 1)/(q - 1) + \varepsilon q^{n/2-1}
\]

with \( \varepsilon = \pm 1 \) if \( n \) is even, and \( \varepsilon = 0 \) if \( n \) is odd.
2 \ n = 3

Suppose first that \( n = 3 \). The parameters \((p^i_{jk})\) were given in BCN p. 375. Let us call them \((a^i_{jk})\) here in the special case \( n = 3 \).

\[
(a^i_{0j})_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad (a^i_{1j})_{ij} = \begin{pmatrix}
0 & \frac{1}{2}q(q-2) & 0 & 0 \\
1 & \frac{1}{4}q(q-2)^2 & \frac{1}{4}q(q-2) & \frac{1}{2}q - 2 \\
0 & \frac{1}{4}q(q-2)^2 & \frac{1}{4}q(q-2) & \frac{1}{2}q - 1 \\
0 & \frac{1}{4}q(q-4) & \frac{1}{4}q^2 & 0 \\
\end{pmatrix},
\]

\[
(a^i_{2j})_{ij} = \begin{pmatrix}
0 & 0 & \frac{1}{2}q^2 & 0 \\
0 & \frac{1}{4}q(q-2) & \frac{1}{4}q^2 & \frac{1}{2}q - 1 \\
0 & \frac{1}{4}q^2 & \frac{1}{4}q^2 & 0 \\
\end{pmatrix}, \quad (a^i_{3j})_{ij} = \begin{pmatrix}
0 & 0 & 0 & q - 2 \\
0 & \frac{1}{2}q^2 & 0 & 0 \\
0 & \frac{1}{2}q - 2 & \frac{1}{2}q - 1 & 0 \\
1 & 0 & 0 & q - 3 \\
\end{pmatrix}.
\]

The \( P \) matrix has in column \( h \) the eigenvalues of \((p^i_{jk})_{ij}\). The rows correspond to eigenspaces. We find

\[
P = \begin{pmatrix}
1 & q(q - 2)/2 & q^2/2 & q - 2 \\
1 & q/2 & -q/2 & -1 \\
1 & -q/2 + 1 & -q/2 & q - 2 \\
1 & -q/2 & q/2 & -1 \\
\end{pmatrix}.
\]

We see that \( R_3 \) is an equivalence relation (and the equivalence classes are the tangent lines, that is, the lines on \( N \)). We also see that \( R_2 \) has only three distinct eigenvalues, and hence defines a strongly regular graph.

Now suppose that \( \dim V = 3 \) but the quadratic form \( Q \) on \( V \) is degenerate in such a way that \( N := V^\perp \) is a (single) isotropic point. Then the space is a cone over a hyperbolic or elliptic line. We have \( v = |X| = q^2 - \varepsilon q \) and the valencies are \( k_0 = 1, \ k_3 = q - 1 \) and \( k_1 = q^2 - 2q, \ k_2 = 0 \) if \( \varepsilon = 1 \), \( k_3 = 0, \ k_2 = q^2 \) if \( \varepsilon = -1 \). Call the corresponding parameters \((h^i_{jk})\) and \((e^i_{jk})\), respectively. Then

\[
(h^i_{1j})_{ij} = \begin{pmatrix}
0 & q^2 - 2q & 0 & 0 \\
1 & q^2 - 3q & 0 & q - 1 \\
* & * & * & * \\
0 & q^2 - 2q & 0 & 0 \\
\end{pmatrix}, \quad (h^i_{3j})_{ij} = \begin{pmatrix}
0 & 0 & 0 & q - 1 \\
0 & q - 1 & 0 & 0 \\
* & * & * & * \\
1 & 0 & 0 & q - 2 \\
\end{pmatrix},
\]

\[
(e^i_{2j})_{ij} = \begin{pmatrix}
0 & 0 & q^2 & 0 \\
* & * & * & * \\
1 & 0 & q^2 - q & q - 1 \\
0 & 0 & q^2 & 0 \\
\end{pmatrix}, \quad (e^i_{3j})_{ij} = \begin{pmatrix}
0 & 0 & 0 & q - 1 \\
* & * & * & * \\
0 & 0 & q - 1 & 0 \\
1 & 0 & 0 & q - 2 \\
\end{pmatrix}.
\]

(with undefined * since relation \( R_2 \) (resp. \( R_1 \)) does not occur).

Finally, suppose that \( \dim V = 3 \) and the quadratic form \( Q \) on \( V \) is a double line (that is, \( B \) vanishes identically, \( Q \) is the square of a linear form). Now \( k_0 = 1, \ k_1 = k_2 = 0, \ k_3 = q^2 - 1 \). Call the corresponding parameters \((z^i_{jk})\). Then

\[
(z^i_{3j})_{ij} = \begin{pmatrix}
0 & 0 & 0 & q^2 - 1 \\
* & * & * & * \\
* & * & * & * \\
1 & 0 & 0 & q^2 - 2 \\
\end{pmatrix}.
\]
3 \textit{n even}

Now let \( n \) be even, say \( n = 2m \), where \( m \geq 2 \). Let the form have type \( \varepsilon \), with \( \varepsilon = 1 \) for a hyperbolic and \( \varepsilon = -1 \) for an elliptic quadric.

The number of points of the scheme equals \( v = |X| = q^{2m-1} - \varepsilon q^{m-1} \).

For the valencies \( k_i \) of the relations \( R_i \), we find

\[
\begin{align*}
k_0 & = 1 \\
k_1 & = (q - 2)q^{m-1}(q^{m-1} + \varepsilon)/2 \\
k_2 & = q^m(q^{m-1} - \varepsilon)/2 \\
k_3 & = q^{2m-2} - 1
\end{align*}
\]

If \( n = 2, m = 1 \), then only one type of lines occurs (since all of \( V \) is just a line), and \( P = \begin{pmatrix} 1 & q - 2 \\ 1 & -1 \end{pmatrix} \) if \( \varepsilon = 1 \), and \( P = \begin{pmatrix} 1 & q \\ 1 & -1 \end{pmatrix} \) if \( \varepsilon = -1 \).

Let \( n \geq 4, m \geq 2 \). If \( (x, y) \in R_h \) for a certain \( h \in \{1, 2, 3\} \) then for each plane on the line \( x + y \) we find the same relation, and a contribution as just computed for the case \( n = 3 \). In the plane we did not count the nucleus, but here that nucleus contributes \( 1 \) to \( p^h_{33} \) for \( h \neq 3 \). If \( h = 3 \) then \( x \) or \( y \) might itself be the nucleus of a nondegenerate plane on \( x + y \). The details follow.

Let \( L \) be a hyperbolic line, and consider the \( (q^{n-2} - 1)/(q - 1) \) planes on \( L \). A degenerate plane must be the span \( L + z \) of \( L \) and a point \( z \) in \( L^\perp \). Now \( L^\perp \) has the same type \( \varepsilon \) as \( V \) and dimension \( n - 2 \), so has \( a := (q^{2m-3} - 1)/(q - 1) + \varepsilon q^{m-2} \) isotropic points. Hence \( L \) is on \( a \) degenerate planes \( L + z \), and on \( (q^{n-2} - 1)/(q - 1) - a = q^{n-3} - \varepsilon q^{m-2} \) nondegenerate planes. All parameters \( p^1_{jk} \) follow by summing such parameters of these two types of planes: If \( (x, y) \in R_1 \), then \( L = x + y \) is a hyperbolic line that contributes \( q - 3 \) to \( p^1_{11} \) and nothing to \( p^1_{jk} \) for \( \{j, k\} \not\subseteq \{0, 1\} \). A degenerate plane on \( L \) is a cone over a hyperbolic line, and contributes \( h^1_{jk} \). Thus

\[
p^1_{11} = q - 3 + (q^{n-3} - \varepsilon q^{m-2})(a^1_{11} - q + 3) + a(h^1_{11} - q + 3)
\]

and

\[
p^1_{33} = (q^{n-3} - \varepsilon q^{m-2})(a^1_{33} + 1) + ah^1_{33}
\]

and

\[
p^1_{jk} = (q^{n-3} - \varepsilon q^{m-2})a^1_{jk} + ah^1_{jk}
\]

for nonzero \( j, k \) not both 1 or both 3.

Let \( L \) be an elliptic line, and consider planes on \( L \). This time \( L^\perp \) has the opposite type, so has \( b := (q^{2m-3} - 1)/(q - 1) - \varepsilon q^{m-2} \) isotropic points, and \( L \) is on \( (q^{n-2} - 1)/(q - 1) - b = q^{n-3} + \varepsilon q^{m-2} \) nondegenerate planes. We find

\[
p^2_{22} = q - 1 + (q^{n-3} + \varepsilon q^{m-2})(a^2_{22} - q + 1) + b(\varepsilon^2_{22} - q + 1)
\]

and

\[
p^2_{33} = (q^{n-3} + \varepsilon q^{m-2})(a^2_{33} + 1) + be^2_{33}
\]

and

\[
p^2_{jk} = (q^{n-3} + \varepsilon q^{m-2})a^2_{jk} + be^2_{jk}
\]

for nonzero \( j, k \) not both 2 or both 3.
Let $L$ be a tangent, with isotropic point $z$. Then $L^\perp$ is an $(n-2)$-space containing $L$. The line $L$ is on $q^{n-3}$ nondegenerate planes (where $Q$ is a conic, $L$ a tangent to the conic, and the nucleus of the plane is a nonisotropic point of $L$), namely those not contained in $z^\perp$. The space $z^\perp / z$ is a nondegenerate $(n-2)$-space of the same type $\varepsilon$ in which $L$ is a nonisotropic point. The quadric in that space has size $(q^{n-3} - 1)/(q - 1) + \varepsilon q^{m-2}$, and through the point $L$ there are $(q^{n-4} - 1)/(q - 1)$ tangents, and $(q^{n-4} + \varepsilon q^{m-2})/2$ hyperbolic lines, and $(q^{n-4} - \varepsilon q^{m-2})/2$ elliptic lines. Consequently, of the $q^{n-4}$ degenerate planes $\pi$ on $L$ with radical $z$, for $(q^{n-4} + \varepsilon q^{m-2})/2$ the quotient $\pi/z$ is hyperbolic, and for $(q^{n-4} - \varepsilon q^{m-2})/2$ elliptic. Each of the $q$ nonisotropic points of $L$ is nucleus of $q^{n-4}$ nondegenerate planes. For the computation of $p^{3}_{jk}$ starting with two points $x, y$ where $L = x + y$ is a tangent, the $q^{n-4}$ nondegenerate planes in which $x$ is nucleus each contribute $\frac{1}{2} q(q-2)$ for $k = 1$ and $\frac{1}{2} q^2$ for $k = 2$. There are $q^{n-4}(q - 2)$ such planes where none of $x, y$ is nucleus. Altogether, we find

$$p^{3}_{jk} = q^{n-4}(q - 2) p^{3}_{jk} + \frac{1}{2} (q^{n-4} + \varepsilon q^{m-2}) h^{3}_{jk} + \frac{1}{2} (q^{n-4} - \varepsilon q^{m-2}) e^{3}_{jk}$$

for $j, k \neq 0, 3$, and

$$p^{3}_{31} = \frac{1}{2} q^{n-3}(q - 2),$$

$$p^{3}_{32} = \frac{1}{2} q^{n-2},$$

$$p^{3}_{33} = q - 2 + \frac{q^n - 1}{q - 1} (z^3_{33} - q + 2).$$

Since we could compute all $p^{i}_{jk}$, this proves that we have an association scheme. Let us substitute the values of $a^{i}_{jk}, h^{i}_{jk}, e^{i}_{jk}$ and $z^{i}_{jk}$ and compute the eigenmatrix $P$ of the scheme. In order to save space, we abbreviate $r := q - 2$.

For $(p^{i}_{jk})_{ij}$ one finds

\[
\begin{pmatrix}
0 & \frac{1}{2} q^{m-1}(q^{m-1} - \varepsilon) r \\
1 & \frac{1}{2} q^{m-3} r^2 + \varepsilon q^{m-2} r (q^2 - 2q - 1) & \frac{1}{2} q^{m-1}(q^{m-1} - \varepsilon) r \\
0 & \frac{1}{2} q^{m-2}(q^{m-1} + \varepsilon) r^2 & \frac{1}{2} q^{m-1}(q^{m-1} - \varepsilon) r \\
0 & \frac{1}{2} q^{m-2}(q^{m-1} - \varepsilon) r & \frac{1}{2} q^{m-2}(q^{m-1} - \varepsilon) r \\
\end{pmatrix}
\]

with eigenvalues $\frac{1}{2} q^{m-1}(q^{m-1} + \varepsilon)(q - 2), \frac{1}{2} \varepsilon q^{m-2}(q + 1)(q - 2), -\varepsilon q^{m-1}, 0$.

For $(p^{3}_{jk})_{ij}$ one finds

\[
\begin{pmatrix}
0 & \frac{1}{2} q^{m-1}(q^{m-1} - \varepsilon) r \\
0 & \frac{1}{2} q^{m-1}(q^{m-1} - \varepsilon) r \\
1 & \frac{1}{2} q^{m-1}(q^{m-1} + \varepsilon) r & \frac{1}{2} q^{m-1}(q^{m-1} - \varepsilon) r \\
0 & \frac{1}{2} q^{m-2}(q^{m-1} + \varepsilon) r & \frac{1}{2} q^{m-1} - \varepsilon q^{m-1}(\frac{1}{2} q - 1) \\
0 & \frac{1}{2} q^{m-2}(q^{m-1} - \varepsilon) r & \frac{1}{2} q^{m-2} - \varepsilon q^{m-1}(\frac{1}{2} q - 1) \\
\end{pmatrix}
\]

with eigenvalues $\frac{1}{2} q^{m}(q^{m-1} - \varepsilon), \varepsilon q^{m-1}, -\frac{1}{2} \varepsilon q^{m-1}(q - 1), 0$.

For $(p^{3}_{jk})_{ij}$ one finds

\[
\begin{pmatrix}
0 & 0 \\
0 & \frac{1}{2} q^{m-2}(q^{m-2} r + 2\varepsilon) \\
0 & \frac{1}{2} q^{m-2}(q^{m-1} + \varepsilon) r & \frac{1}{2} q^{m-1} - \varepsilon q^{m-1}(\frac{1}{2} q - 1) \\
1 & \frac{1}{2} q^{m-3} r \\
\end{pmatrix}
\]

with eigenvalues $\frac{1}{2} q^{m-2}(q^{m-2} - \varepsilon), \varepsilon q^{m-2}(q^{m-1} - \varepsilon), q^{m-2}(q^{m-1} + \varepsilon), \frac{1}{2} q^{m-3} - 2$.
with eigenvalues \( q^{n-2} - 1, q^{m-1} - 1, -q^{m-1} - 1, \varepsilon q^{m-2} - 1 \).

The \( P \)-matrix is

\[
P = \begin{pmatrix}
1 & \frac{1}{2} q^{m-1}(q^{m-1} + \varepsilon)(q - 2) & \frac{1}{2} q^{m}(q^{m-1} - \varepsilon) & q^{2m-2} - 1 \\
1 & \frac{1}{2} \varepsilon q^{m-2}(q + 1)(q - 2) & -\frac{1}{2} \varepsilon q^{m-1}(q - 1) & \varepsilon q^{m-2} - 1 \\
1 & 0 & \varepsilon q^{m-1} - 1 & \varepsilon q^{m-1} - 1 \\
1 & 0 & -\varepsilon q^{m-1} & 0
\end{pmatrix}.
\]

The multiplicities (in the order of the rows of \( P \)) are \( 1, \frac{1}{2} q(q^{m-1} - \varepsilon)(q^m - \varepsilon)/(q + 1), \frac{1}{2}(q - 2)(q^{m-1} + \varepsilon)(q^m - \varepsilon)/(q - 1) \).

### 4 n odd

Now let \( n \) be even, say \( n = 2m + 1 \), where \( m \geq 2 \). Let \( Q \) be a nondegenerate quadric, and let \( N \) be its nucleus. We compute the \( p^i_{jk} \) as before, this time splitting relation \( R_3 \) (being joined by a tangent) into the two relations \( R_{3a} \) and \( R_{3n} \), depending on whether the tangent does not or does pass through \( N \).

The number of points of the scheme equals \( v = |X| = q^{n-1} - 1 \).

For the valencies \( k_i \) of the relations \( R_i \) we find

\[
\begin{align*}
k_0 &= 1 \\
k_1 &= \frac{1}{2} q^{n-2}(q - 2) \\
k_2 &= \frac{1}{2} q^{n-1} \\
k_{3a} &= q^{n-2} - q \\
k_{3n} &= q - 2
\end{align*}
\]

The number of planes on a line \( L \) is \( (q^{n-2} - 1)/(q - 1) \). If \( L \) is hyperbolic or elliptic, then a degenerate plane must be the span \( L + z \) of \( L \) and an isotropic point \( z \) in \( L^1 \). Now \( L^1 \) is a nondegenerate \((n-2)\)-space, and has \((q^{n-3} - 1)/(q - 1)\) isotropic points, so there are \( q^{n-3} \) nondegenerate planes, and \((q^{n-3} - 1)/(q - 1)\) degenerate planes on \( L \). We find for \( i = 1, 2 \) that

\[
p^i_{jk} = q^{n-3}(a^i_{jk} - c) + \frac{q^{n-3} - 1}{q - 1}(x^i_{jk} - c) + c
\]

with \( x = h \) for \( i = 1 \) and \( x = e \) for \( i = 2 \), and \( c = q - 3 \) if \( i = j = k = 1 \), \( c = q - 1 \) if \( i = j \) or \( k = 2 \) and \( c = 0 \) otherwise.

If \( L \) is a tangent on \( N \), with isotropic point \( z \), then the \( q^{n-3} \) nondegenerate planes on \( L \) are the planes not in \( z^1 \). The remaining \((q^{n-3} - 1)/(q - 1)\) planes on \( L \) are contained in \( L^1 \), and the form induces a double line on these. Hence

\[
p^i_{jk} = q^{n-3} a^3_{jk}
\]

for \( i = 3n \) when not \( \{j, k\} \subseteq \{0, 3a, 3n\} \).

If \( L \) is a tangent not on \( N \), with isotropic point \( z \), then the \( q^{n-3} \) nondegenerate planes on \( L \) are the planes not in \( z^1 \). Each nonsotropic point of \( L \) is the nucleus of \( q^{n-4} \) of these planes. There are \((q^{n-4} - 1)/(q - 1)\) planes on \( L \) contained in \( L^1 \), where the form induces a double line. The remaining planes are degenerate, cones over a hyperbolic or elliptic line, \( \frac{1}{2} q^{n-4} \) of each.
Relation $R_{3n}$ is an equivalence relation with equivalence classes of size $q - 1$. If $L$ does not pass through $N$, then it is on a unique plane $L + N$ on $N$, and the points that have relation $R_{4n}$ with $x$ or $y$ live in that plane. We find $p_{1,3n}^1 = \frac{1}{2} q - 2$, $p_{2,3n}^1 = \frac{1}{2} q$, $p_{1,3n}^2 = p_{2,3n}^2 = \frac{1}{2} q - 1$.

For $(p_{1j}^i)$ one finds

$$
\begin{pmatrix}
0 & \frac{1}{2}q^{n-2}(q-2) & 0 & 0 & 0 \\
1 & \frac{1}{2}q^{n-3}(q-2)^2 & \frac{1}{2}q^{n-2}(q-2) & \frac{1}{2}(q^{n-3} - 1)(q-2) & \frac{1}{2} q - 2 \\
0 & \frac{1}{4}q^{n-3}(q-2)^2 & \frac{1}{2}q^{n-2}(q-2) & \frac{1}{2}(q^{n-3} - 1)(q-2) & \frac{1}{2} q - 1 \\
0 & \frac{1}{4}q^{n-3}(q-2)^2 & \frac{1}{2}q^{n-2}(q-2) & \frac{1}{2}q^{n-3}(q-2) & 0 \\
0 & \frac{1}{4}q^{n-2}(q-4) & \frac{1}{2}q^{n-1} & 0 & 0
\end{pmatrix}
$$

with eigenvalues $\frac{1}{2} q^{2m-1}(q-2)$, $\pm \frac{1}{2} q^{m-1}(q-2)$, $\pm \frac{1}{2} q^m$.

For $(p_{2j}^i)$ one finds

$$
\begin{pmatrix}
0 & 0 & \frac{1}{2}q^{n-1} & 0 & 0 \\
0 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}q^{n-1} & \frac{1}{2} q(q^{n-3} - 1) & \frac{1}{2} q \\
1 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}q^{n-1} & \frac{1}{2} q(q^{n-3} - 1) & \frac{1}{2} q + 1 \\
0 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}q^{n-1} & \frac{1}{2}q^{n-2} & 0 \\
0 & \frac{1}{4}q^{n-2}(q-2) & \frac{1}{2}q^{n-1} & 0 & 0
\end{pmatrix}
$$

with eigenvalues $\frac{1}{2} q^m$, $\pm \frac{1}{2} q^m$ (each twice).

For $(p_{3a,j}^i)$ one finds

$$
\begin{pmatrix}
0 & 0 & 0 & \frac{1}{2}(q^{n-3} - 1)(q-2) & 0 \\
0 & \frac{1}{2}(q^{n-3} - 1)(q-2) & \frac{1}{2} q(q^{n-3} - 1) & q^{n-3} - 1 & 0 \\
0 & \frac{1}{2}(q^{n-3} - 1)(q-2) & \frac{1}{2} q(q^{n-3} - 1) & q^{n-3} - 1 & 0 \\
1 & \frac{1}{2}q^{n-3}(q-2) & \frac{1}{2}q^{n-2} & q^{n-3} - 2q + 1 & q - 2 \\
0 & 0 & 0 & q(q^{n-3} - 1) & 0
\end{pmatrix}
$$

with eigenvalues $q(q^{2m-2} - 1)$, $(q^{m-1} - 1)(q-1)$, $-(q^{m-1} + 1)(q-1)$, 0 (twice).

For $(p_{3n,j}^i)$ one finds

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & q - 2 \\
0 & \frac{1}{2} q^2 & \frac{1}{2} q & 0 & 0 \\
0 & \frac{1}{2} q - 1 & \frac{1}{2} q - 1 & 0 & 0 \\
0 & 0 & 0 & q - 2 & 0 \\
1 & 0 & 0 & 0 & q - 3
\end{pmatrix}
$$

with eigenvalues $q - 2$ (three times) and $-1$ (twice).

Since we could compute all $p_{jk}^i$, this is indeed an association scheme.

The $P$-matrix is

$$
P = \begin{pmatrix}
1 & \frac{1}{2} q^{2m-1}(q-2) & \frac{1}{2} q^{2m} & q(q^{2m-2} - 1) & q - 2 \\
1 & \frac{1}{2} q^{n-1}(q-2) & \frac{1}{2} q^m & -(q^{m-1} + 1)(q-1) & q - 2 \\
1 & -\frac{1}{2} q^{m-1}(q-2) & -\frac{1}{2} q^m & (q^{m-1} - 1)(q-1) & q - 2 \\
1 & \frac{1}{2} q^m & -\frac{1}{2} q^m & 0 & -1 \\
1 & -\frac{1}{2} q^m & \frac{1}{2} q^m & 0 & -1
\end{pmatrix}
$$

The multiplicities (in the order of the rows of $P$) are 1, $\frac{1}{2} q(q^m + 1)(q^{m-1} - 1)/(q-1)$, $\frac{1}{2} q(q^m - 1)(q^{m-1} + 1)/(q-1)$, $\frac{1}{2}(q - 2)(q^{2m - 1} - 1)/(q-1)$ (twice).
5 Conclusion

Vanhove computed all $p_{jk}$ and communicated both $P$ matrices. We recomputed the $p'_{jk}$ and the $P$ matrices and find the same results.

References
