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A NOTE ON ΓΔ-REGULAR GRAPHS

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A note on ΓΔ-regular graphs

by

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dedicated to M. Voorhoeve & R. Tijdeman

ABSTRACT

A ΓΔ-regular graph is a nonregular graph such that for each vertex x the graphs induced on its neighbours and on its nonneighbours are both regular. We show that if G is ΓΔ-regular, G and $\bar{G}$ are connected, and diam G = 3 then G is one of two graphs on 4 resp. 8 vertices.

GODSIL & MCKAY [1] introduced the concept of a ΓΔ-regular graph (although they called it differently - we adopt Van Lint's terminology) - see the abstract. For the case we are interested in: G and $\bar{G}$ are connected, they proved the following.

Let n be the number of vertices, λ the valency of $\Gamma(x)$ in G and $\bar{\lambda}$ the valency of $\bar{\Gamma}(x)$ in $\bar{G}$.

(1) In G there occur exactly two valencies, k_1 and k_2, where $k_1 < k_2$.

(2) $k_1 + k_2 = \frac{1}{2}n + 2\lambda + 1$

(3) $\lambda + \bar{\lambda} = \frac{1}{2}n - 2$

Let $M_i = \{x \mid x \text{ has valency } k_i\}$ $(i = 1,2)$. Write $m_i := |M_i|$.

(4) Each M_i (viewed as induced subgraph of G) is regular with valency α_i $(i = 1,2)$, $\alpha_1 + \alpha_2 = \frac{1}{2}n - 1$, $(2\alpha_i - m_i + 1)(k_1 - k_2) = (\lambda+1)(n-1) - k_1 k_2$.

(5) Let $x_1 \neq x_2$. Then $|\Gamma(x_1) \cap \Gamma(x_2)| = \lambda + 1 + \varepsilon(k_1 - k_2)$, where

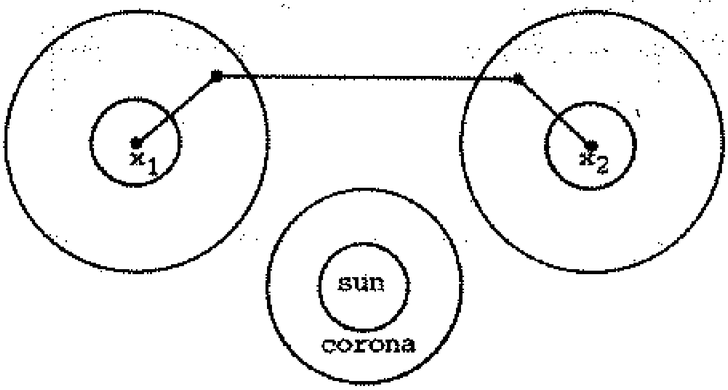
$\varepsilon = 0$ if $x_1 \in M_1$, $x_2 \in M_2$

$\varepsilon = 1$ if $x_1, x_2 \in M_1$,

$\varepsilon = -1$ if $x_1, x_2 \in M_2$.

(6) diam $G \leq 3$.

Now suppose G has diameter 3, and let $\text{dist}(x_1, x_2) = 3$. Then $\Gamma(x_1) \cap \Gamma(x_2) = \emptyset$ so that by (5) $x_1, x_2 \in M_1$ and $k_2 - k_1 = \lambda + 1$. Again by (5) points in M_1 do not have distance two, so that M_1 is a disjoint union of cliques ('sun's). Also, no point of M_2 is adjacent to points of different suns but each point of M_2 is adjacent to some point in M_1 (in fact to $k_2 - \alpha_2$ such points; $k_2 - \alpha_2 > 0$ since G is connected), so that the partition of M_1 into suns induces a partition of M_2 into 'corona's. From (2) and $k_2 - k_1 = \lambda + 1$ we find $n = 4k_1 - 2\lambda$. On the other hand, choosing one vertex in each sun we find $n \geq (k_1+1)$. # of suns. Consequently the number of suns N is less than four (and larger than one since diam G = 3), i.e. two or three.



Fix a point $x_0 \in M_1$ and count edges between $\Gamma(x_0)$ and $\Delta(x_0)$. One finds

$$\alpha_1(k_1-\lambda-1)+(k_1-\alpha_1)(k_2-\lambda-1) = (n-k_1-1-(N-1)(\alpha_1+1))(k_2-k_1)$$

(for: $|\Gamma(x_0)| = k_1$, $|\Delta(x_0)| = n - k_1 - 1$, $|\mathrm{sun}| = \alpha_1 + 1$, etc.), i.e.,

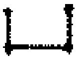
$$k_1^2 - \alpha_1(\lambda+1) = (3k_1-2\lambda-1-(N-1)(\alpha_1+1))(\lambda+1),$$

or

$$k_1^2 - 3(\lambda+1)k_1 + (\lambda+1)(2\lambda+2+(N-2)(\alpha_1+1)) = 0.$$

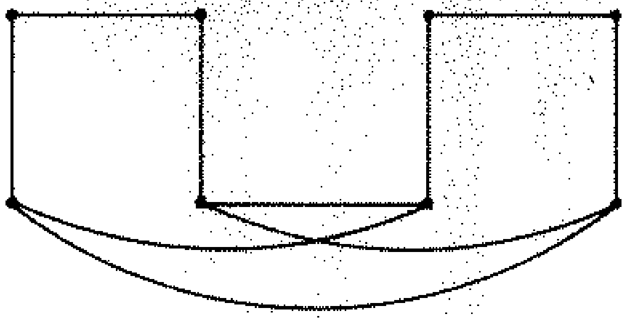
Distinguish cases:

A. If $N = 2$ this factors as $(k_1-(\lambda+1))(k_1-2(\lambda+1)) = 0$.

A1. $N = 2$ and $k_1 = \lambda + 1$.
Now $k_2 = 2\lambda + 2$, $n = 2\lambda + 4$, i.e. $|\mathrm{sun}| = 1$, $|\mathrm{corona}| = \lambda + 1$. Considering two adjacent points in different coronas we find that they have 2λ common neighbours. Hence $\lambda = 2\lambda$, i.e. $\lambda = 0$, $n = 4$ and G looks like .

A2. $N = 2$ and $k_1 = 2(\lambda+1)$.
Now $k_2 = 3\lambda + 3$, $n = 6\lambda + 8$, $|\mathrm{sun}| = \alpha_1 + 1$, $|\mathrm{corona}| = 3\lambda + 3 - \alpha_1$. Count edges between sun and corona: $(\alpha_1+1)(k_1-\alpha_1) = (3\lambda+3-\alpha_1)(k_2-\alpha_2)$, but $k_2 - \alpha_2 = 2\lambda + 2 - k_1 + \alpha_1 = \alpha_1$ (using (2) and (4)), so that

$\alpha_1 = 1 + \frac{\lambda}{\lambda+2}$. Since α_1 is integral this implies $\lambda = 0$, $n = 8$ and G looks like



B. If $N = 3$ then $|\text{sun} + \text{corona}| = \frac{4}{3}k_1 - \frac{2}{3}\lambda = 1 + k_1 + \frac{1}{3}(k_1-2\lambda-3)$.

As before it follows that $k_1 \geq 2\lambda + 3$, contradicting the equation

$$k_1^2 - 3(\lambda+1)k_1 + (\lambda+1)(2\lambda+\alpha_1+3) = 0.$$

This ends the proof.

Egeldonk, 80 09 25

REFERENCE

[1] GODSIL, C.D. & B.D. MCKAY, *Graphs with regular neighbourhoods*, to appear in: proceedings of Australian combinatorial conference, (Newcastle, 1979).