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ON THE PACKING OF QUADRUPLES WITHOUT COMMON TRIPLES

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On the packing of quadruples without common triples *)

by

A.E. Brouwer

ABSTRACT

We observe that the packing number d(3,4,6m) can be readily inferred from results available in the literature.

KEY WORDS & PHRASES: *packing*, *quadruple system*

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*) This report will be submitted for publication elsewhere

INTRODUCTION

Let d(t,k,v) denote the maximum cardinality of a family of k-subsets of a v-set such that no two of these k-sets have a t-set in common. JOHNSON [7] showed that $d(t,k,v) \le [\frac{v}{k}d(t-1,k-1,v-1)]$ and $d(t,k,v) \le [\frac{v-k}{k}d(t,k,v-1)]$. The first inequality is well known (and usually ascribed to SCHÖNHEIM [10]) but the second one is often reproved for special cases by some ad hoc arguments. Usually d(t,k,v) equals the upper bound obtained by repeatedly applying these inequalities; I conjecture that given t and k there are only finitely many values of v for which d(t,k,v) does not equal the Johnson bound. SCHÖNHEIM [10] determined d(2,3,v) showing that it always equals the Johnson bound:

$$d(2,3,v) = \begin{cases} [\frac{v}{3}[\frac{v-1}{2}]] & \text{if } v \not\equiv 5 \pmod 6 \\ [\frac{v}{3}[\frac{v-1}{2}]]-1 & \text{if } v \equiv 5 \pmod 6 \end{cases}$$

In BROUWER [3] it is shown that d(2,4,v) equals the Johnson bound iff

$v \ne 8,9,10,11,17,19$.

Concerning d(3,4,v) the following is known:

(i) for $v \equiv 2$ or $4 \pmod 6$ HANANI [5] constructed a Steiner system S(3,4,v) proving that $d(3,4,v) = v(v-1)(v-2)/24$.

(ii) from this and the results of SCHÖNHEIM [10] it readily follows that for $v \equiv 1$ or $3 \pmod 6$ we have $d(3,4,v) = v(v-1)(v-3)/24$.

(iii) for $v \equiv 0 \pmod 6$ I announced in [2] that $d(3,4,v) = [\frac{v}{4}\cdot d(2,3,v-1)] = v(v^2-3v-6)/24$.
[The case where $v = 6\cdot 2^n$ was treated in KALBFLEISCH & STANTON [8].] The main purpose of this note is to show that this equality is a trivial consequence of results already available in the literature.

(iv) for $v \equiv 5 \pmod 6$ almost nothing is known. Of course d(3,4,5) = 1, and BEST ([1], see also [2]) showed that d(3,4,11) = 35. Finally there is some information on the structure of a system meeting the Johnson bound.

(Note that in all known cases d(3.4.v) equals the Johnson bound.)

THE CASE $v = 6m$

Let $X = I_6 \times Y$ where $I_6 = \{0,1,2,3,4,5\}$ and $|Y| = m$.

MILLS [9] defines a $G(m,6,4,3)$ system to be a collection $\mathcal{B}$ of 4-subsets of X such that every triple in X that is not contained in $I_6 \times \{y\}$ for some $y \in Y$ is contained in exactly one 4-set $B \in \mathcal{B}$. He proceeds to show ([9], theorem 1) that such a system exists for all m.

[This can be reformulated by saying that for each m there exists a triplewise balanced design TBD({4,6},1;6m) such that the blocks of size 6 form a parallelclass.]

Now $$|G(m,6,4,3)| = \frac{\binom{6m}{3} - m\binom{6}{3}}{\binom{4}{3}} = \frac{v(v-1)(v-2)}{24} - 5m.$$

Adding to a system $G(m,6,4,3)$ the 3m quadruples

$$\{0,1,2,3\} \times \{y\},\ \{0,1,4,5\} \times \{y\},\ \{2,3,4,5\} \times \{y\} \qquad (y \in Y)$$

yields a packing $D(3,4,v)$ containing $\frac{v(v-1)(v-2)}{24} - 2m = v(v^2-3v-6)/24$ quadruples. Since this is the Johnson bound for this case it follows that

$$d(3,4,v) = v(v^2-3v-6)/24$$

for $v \equiv 0 (\mathrm{mod}\, 6)$.

REMARK

KALBFLEISCH & STANTON [8] showed conversely that for any system $D(3,4,v)$ with $v(v^2-3v-6)/24$ quadruples (where $v = 6m$) it is possible to partition the underlying set X into m sets of six elements such that each of the 8m non-covered triples is contained in one of these 6-sets, and each 6-set contains 8 non-covered triples. But this is the same as saying that each optimal $D(3,4,v)$ system (with $v = 6m$) can be obtained from a $G(m,6,4,3)$ system in the way described above.

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