AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER
ON THE PACKING OF QUADRUPLES WITHOUT COMMON TRIPLES

Preprint

2e boerhaavestraat 49 amsterdam
Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).
On the packing of quadruples without common triples *)

by

A.E. Brouwer

ABSTRACT

We observe that the packing number \( d(3,4,6m) \) can be readily inferred from results available in the literature.

KEY WORDS & PHRASES: packing, quadruple system

*) This report will be submitted for publication elsewhere
INTRODUCTION

Let \( d(t,k,v) \) denote the maximum cardinality of a family of \( k \)-subsets of a \( v \)-set such that no two of these \( k \)-sets have a \( t \)-set in common. JOHNSON [7] showed that \( d(t,k,v) \leq \frac{v}{k} d(t-1,k-1,v-1) \) and \( d(t,k,v) \leq \left\lfloor \frac{v-k}{k} d(t,k,v-1) \right\rfloor \). The first inequality is well known (and usually ascribed to SCHÖNHEIM [10]) but the second one is often reproved for special cases by some ad hoc arguments. Usually \( d(t,k,v) \) equals the upper bound obtained by repeatedly applying these inequalities; I conjecture that given \( t \) and \( k \) there are only finitely many values of \( v \) for which \( d(t,k,v) \) does not equal the Johnson bound. SCHÖNHEIM [10] determined \( d(2,3,v) \) showing that it always equals the Johnson bound:

\[
d(2,3,v) = \begin{cases} 
\left\lfloor \frac{v(v-1)}{3} \right\rfloor & \text{if } v \not\equiv 5 \pmod{6} \\
\left\lfloor \frac{v(v-1)}{3} \right\rfloor - 1 & \text{if } v \equiv 5 \pmod{6}
\end{cases}
\]

In BROWNER [3] it is shown that \( d(2,4,v) \) equals the Johnson bound iff \( v \not\equiv 8,9,10,11,17,19 \).

Concerning \( d(3,4,v) \) the following is known:

(i) for \( v \equiv 2 \) or \( 4 \pmod{6} \) HANANI [5] constructed a Steiner system \( S(3,4,v) \) proving that \( d(3,4,v) = v(v-1)(v-2)/24 \).

(ii) from this and the results of SCHÖNHEIM [10] it readily follows that for \( v \equiv 1 \) or \( 3 \pmod{6} \) we have \( d(3,4,v) = v(v-1)(v-3)/24 \).

(iii) for \( v \equiv 0 \pmod{6} \) I announced in [2] that \( d(3,4,v) = \left\lfloor \frac{v}{4} d(2,3,v-1) \right\rfloor = v(v^2 - 3v - 6)/24 \).

[The case where \( v = 6 \cdot 2^n \) was treated in KALBFLEISCH & STANTON [8].] The main purpose of this note is to show that this equality is a trivial consequence of results already available in the literature.

(iv) for \( v \equiv 5 \pmod{6} \) almost nothing is known. Of course \( d(3,4,5) = 1 \), and BEST ([1], see also [2]) showed that \( d(3,4,11) = 35 \). Finally there is some information on the structure of a system meeting the Johnson bound.

(Note that in all known cases \( d(3,4,v) \) equals the Johnson bound.)
THE CASE $v = 6m$

Let $X = I_6 \times Y$ where $I_6 = \{0,1,2,3,4,5\}$ and $|Y| = m$.

MILLS [9] defines a $G(m,6,4,3)$ system to be a collection $B$ of 4-subsets of $X$ such that every triple in $X$ that is not contained in $I_6 \times \{y\}$ for some $y \in Y$ is contained in exactly one 4-set $B \in B$. He proceeds to show ([9], theorem 1) that such a system exists for all $m$.

[This can be reformulated by saying that for each $n$ there exists a triplewise balanced design TBD($\{4,6\}$,$1;6m$) such that the blocks of size 6 form a parallelclass.]

Now

$$|G(m,6,4,3)| = \frac{\binom{6m}{3} - m\binom{6}{3}}{\binom{4}{3}} = \frac{v(v-1)(v-2)}{24} = 5m.$$

Adding to a system $G(m,6,4,3)$ the $3m$ quadruples

$\{0,1,2,3\} \times \{y\}$, $\{0,1,4,5\} \times \{y\}$, $\{2,3,4,5\} \times \{y\}$ ($y \in Y$)

yields a packing $D(3,4,v)$ containing $\frac{v(v-1)(v-2)}{24} - 2m = v(v^2 - 3v - 6)/24$ quadruples. Since this is the Johnson bound for this case it follows that

$$d(3,4,v) = \frac{v(v^2 - 3v - 6)}{24}$$

for $v = 0(mod 6)$.

REMARK

KALBFLEISCH & STANTON [8] showed conversely that for any system $D(3,4,v)$ with $v(v^2 - 3v - 6)/24$ quadruples (where $v = 6m$) it is possible to partition the underlying set $X$ into $m$ sets of six elements such that each of the $8m$ non-covered triples is contained in one of these 6-sets, and each 6-set contains 3 non-covered triples. But this is the same as saying that each optimal $D(3,4,v)$ system (with $v = 6m$) can be obtained from a $G(m,6,4,3)$ system in the way described above.
REFERENCES


Tel Aviv, 771020