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AFDELING ZUIVERE WISKUNDE ZW 146/80 NOVEMBER
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER & A.W.J. KOLEN

A SUPER-BALANCED HYPERGRAPH HAS A NEST POINT

kruislaan 413 1098 SJ amsterdam

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Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

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1980 Mathematics subject classification: 05C65

A super-balanced hypergraph has a nest point

by

Andries Brouwer & Antoon Kolen
Mathematisch Centrum,
Amsterdam

ABSTRACT

A super-balanced hypergraph is a hypergraph such that in any cycle of length at least three there is an edge containing at least three vertices of the cycle. A nest point is a vertex such that the edges containing it are totally ordered by inclusion. It is proved that a super-balanced hypergraph contains at least two nest points.

KEY WORDS & PHRASES: *hypergraphs, balanced hypergraphs, chordal graphs*

1. INTRODUCTION

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, and let $E = \{E_i | i = 1,2,\ldots,m\}$ be a family of subsets of X. The family E is said to be a *hypergraph on* X if

(1) $E_i \neq \emptyset \quad i = 1,2,\ldots,m,$

(2) $\bigcup_{i=1}^{m} E_i = X$

The couple $H = (X, E)$ is called a *hypergraph*. The elements $x_1, x_2, \ldots, x_n$ are called *vertices* and the sets $E_1, E_2, \ldots, E_m$ are called *edges*.
The *dual hypergraph* $H^* = (X^* E^*)$ is defined by $X^* = \{e_1, \ldots, e_m\}$, $E^* = \{X_1, X_2, \ldots, X_n\}$, where $X_j = \{e_i | x_j \in E_i,\ i = 1,\ldots,m\}$, $j = 1,2,\ldots,n$.
In a hypergraph $H = (X, E)$, *a chain of length* q is defined to be a sequence $(x_1, E_1, x_2, E_2, \ldots, E_q, x_{q+1})$ such that
(1) $x_1, x_2, \ldots, x_q$ are all distinct vertices of H,
(2) $E_1, E_2, \ldots, E_q$ are all distinct edges of H,
(3) $x_k, x_{k+1} \in E_k$ for $k = 1,2,\ldots,q$.
If $q > 1$ and $x_{q+1} = x_1$, then this chain is called a *cycle of length* q.
A hypergraph H is said to be *balanced* if every odd cycle has an edge that contains at least three vertices of the cycle. Balanced hypergraphs have been studied quite extensively (BERGE [1,2]). They have the following property.

<u>PROPERTY 1.1</u> Let $J \subseteq \{1,2,\ldots,m\}$ be such that if $E_i \cap E_j \neq \emptyset$ for all $i,j \in J$, then $\bigcap_{i \in J} E_i \neq \emptyset$, i.e., the edges of a balanced hypergraph have Helly's property.

We consider a more restrictive class of hypergraphs called super-balanced. A hypergraph is said to be *super-balanced* if every cycle of length at least three has an edge containing at least three vertices of the cycle.

<u>PROPERTY 1.2</u>. The dual hypergraph H^* of a super-balanced hypergraph H is super-balanced.

<u>PROOF</u>. Consider a cycle $\mu = (e_1, X_1, e_2, \ldots, X_p, e_1)$. It corresponds to a cycle $(x_1, E_2, e_2, \ldots, x_p, E_1, x_1)$ in H.
As H is super-balanced an edge E_i contains three of the x_j's and therefore in H^* an e_i belongs to three of the X_j's; or equivalently an X_k contains three of the e_i's. □

As an example of a super-balanced hypergraph we consider the following.

EXAMPLE. Let T be a tree with vertex set $V = \{v_1,\ldots,v_p\}$. Each edge of the tree has a positive length. The distance $d(v_i,v_j)$ between two vertices $v_i,v_j \in V$ is defined to be the length of the shortest path between these two vertices. For each i, $1 \le i \le p$, let r_i be a nonnegative integer and define $E_i = \{v \in V | d(v,v_i) \le r_i\}$. It was shown by GILES [4] that the hypergraph $(V,\{E_1,\ldots,E_p\})$ is super-balanced.

A *subhypergraph* of (X,E) is a hypergraph (A,E^A), where $A \subset X$ and $E^A = \{E_i \cap A \mid E_i \in E, E_i \cap A \neq \emptyset\}$.
A *partial hypergraph* of (X,E) is a hypergraph (X_F,F), where $F \subset E$ and $X_F = \bigcup_{E_i \in F} E_i$.
The following properties of super-balanced hypergraphs are trivial.

PROPERTY 1.3. If H is a super-balanced hypergraph, then every partial hypergraph H' is super-balanced.

PROPERTY 1.4. If H is a super-balanced hypergraph, then every subhypergraph H' is super-balanced.

A *nest point* of a hypergraph is a vertex with the property that the edges containing it are totally ordered by inclusion.
The *incidence matrix* $A = (a_{ij})$ of a hypergraph (X,E) is defined by $a_{ij} = 1$ if $x_j \in E_i$, $a_{ij} = 0$ otherwise.
A (0,1)-matrix is called *super-balanced* if it does not contain a square submatrix of size at least three with row and column sums equal to two. It is clear that the definition of a super-balanced hypergraph that the incidence matrix of a super-balanced hypergraph is super-balanced. The converse is trivially true: every super-balanced matrix defines a super-balanced hypergraph. Our interest in proving that a super-balanced hypergraph has a nest point arises from the fact that this result enables us to solve the set covering problem on a super-balanced matrix in polynomial time; in a subsequent paper we will show how this is done.
The vertex *intersection graph* $G = (X,\Gamma)$ corresponding to a hypergraph (X,E) has vertex set X and two vertices are adjacent if and only if they have an edge of E in common.

A *chordal graph* is a graph with the property that every cycle with more than three vertices has a chord, i.e., an edge incident to two vertices of the cycle which are not incident to an edge of the cycle.
The following porperty follows from the definition of a super-balanced hypergraph.

PROPERTY 1.5. The vertex intersection graph of a super-balanced hypergraph is a chordal graph.

Chordal graphs are sometimes called *rigid circuit graphs* or *triangulated graphs*.
A *simplicial vertex* of a graph is a vertex with the property that if two vertices are adjacent to this vertex, then they are also adjacent to each other, i.e., all vertices adjacent to a simplicial vertex form a clique.
The following property of a chordal graphs was first proved by DIRAC [3].

PROPERTY 1.6. A chordal graph has a simplicial vertex.

2. MAIN RESULT.

In this section we will prove our main result, namely that every super-balanced hypergraph (X,E) with at least two vertices contains at least two nest points. To prove this result we will use induction on the number $|X| + |E|$. It is clear that if $|X| = 2$, then both vertices are nest points. Consider a super-balanced hypergraph (X,E) and assume that all super-balanced hypergraphs $(\hat{X},\hat{E})$ with $|\hat{X}| + |\hat{E}| < |X| + |E|$ have at least two nest points. In particular this is the case for all partial subhypergraphs of (X,E) (by Property 1.3 and 1.4). Our result follows from the next two theorems.

THEOREM 2.1. *The super-balanced hypergraph* (X,E) *has a nest point.*

THEOREM 2.2. *The super-balanced hypergraph* (X,E) *does not contain exactly one nest point.*

Before proving these theorems we give some definitions and prove some useful lemmas.

DEFINITIONS.
- $\mathcal{E}_x = \{E \mid E \in \mathcal{E},\ x \in E\}$, for $x \in X$.
- $\mathcal{E}(A) = \{E\backslash A \mid E \in \mathcal{E}\}$, for $A \subset X$.
- Two edges E_1 and E_2 are *comparable* if $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$
- Two edges are *incomparable* if they are not comparable.
- Let $\mathcal{E}_x$ $(x \in X)$ be totally ordered by inclusion. Then *min* $\mathcal{E}_x$ is an edge belonging to $\mathcal{E}_x$ with the property that it is included in all edges of $\mathcal{E}_x$, *max* $\mathcal{E}_x$ is the edge belonging to $\mathcal{E}_x$ which includes all other edges of $\mathcal{E}_x$.

LEMMA 2.3. *Let* $E,F_1,F_2 \in \mathcal{E}$ *such that* $E \subseteq F_1 \cap F_2$ *and* F_1 *and* F_2 *incomparable. Then* $(X,\mathcal{E})$ *contains two nest points.*

PROOF. Consider the partial hypergraph obtained from $(X,\mathcal{E})$ by deleting the edge E. By induction this hypergraph has two nest points y_1 and y_2. Since F_1 and F_2 are incomparable it follows that $y_i \notin E \subseteq F_1 \cap F_2$ $(i = 1,2)$. Hence y_1 and y_2 are also nest points of $(X,\mathcal{E})$. □

According to Lemma 2.3. we may assume without loss of generality that $(X,\mathcal{E})$ satisfies Property 2.4.

PROPERTY 2.4. If $E,F_1,F_2 \in \mathcal{E}$ and $E \subseteq F_i$ $(i=1,2)$, then F_1 and F_2 are comparable.

LEMMA 2.5. *Let* $(X,\mathcal{E})$ *be a super-balanced hypergraph. Then there is a vertex* x_1 *and an edge* E_1 *with* $x_1 \in E_1$ *such that* $\forall F \in \mathcal{E}_{x_1} [F \subseteq E_1]$.

PROOF. Consider the vertex intersection graph G of $(X,\mathcal{E})$, By Properties 1.5 and 1.6 there is a simplicial vertex x_1 of G. Let $x_2,\dots,x_k$ be all vertices adjacent to x_1. Clearly all edges belonging to $\mathcal{E}_{x_1}$ are included in $\{x_1,x_2,\dots,x_k\}$. Consider the dual hypergraph of $(X,\mathcal{E})$. Since $x_1,x_2,\dots,x_k$ form a clique in G we know that for the corresponding edges $X_1,X_2,\dots,X_k$ of the dual hypergraph $X_i \cap X_j \neq \emptyset$ for all $i,j = 1,\dots,k$. By Property 1.1 we known that there is a vertex e_1 of the dual hypergraph such that $e_1 \in \bigcap_{i=1}^{k} X_i$. Therefore the corresponding edge E_1 of $(X,\mathcal{E})$ contains $x_1,x_2,\dots,x_k$. Since all edges containing x_1 are contained in $\{x_1,x_2,\dots,x_k\}$ we know that $E_1 = \{x_1,x_2,\dots,x_k\}$. □

PROOF OF THEOREM 2.1. Let x_1, E_1 be the vertex and edge found in Lemma 2.5. Define the set I by $I = \{i \in E_1 \mid \forall F \in \mathcal{E}_i [F \subseteq E_1]\}$. Since $x_1 \in I$ we know that $I \neq \emptyset$. We consider two possibilities.

1. $I = E_1$.
   Consider the partial hypergraph obtained from $(X,\mathcal{E})$ by deleting E_1. By induction it contains two nest points y_1 and y_2. If $y_i \notin E_1$, then y_i is nest point of $(X,\mathcal{E})$ $(i = 1,2)$. If $y_i \in E_1 = I$, then by definition of I all edges of $\mathcal{E}_{y_i}$ are included in E_1, hence y_i is also a nest point of $(X,\mathcal{E})$ $(i = 1,2)$.

2. $I \subsetneq E_1$.
   Consider the subhypergraph of $(X,\mathcal{E})$ obtained by deleting the set I from X. Let $\hat{\mathcal{E}} = \mathcal{E}(I)$. Let y be a nest point of this hypergraph. If there is no edge belonging to $\mathcal{E}$ which contains y and a point $i \in I$, then y is nest point of $(X,\mathcal{E})$. So we may assume that there is an edge $F \in \mathcal{E}$ containing both y and a point $i \in I$. By definition of I we know that $F \subseteq E_1$ and therefore $y \in E_1 \backslash I$. Since $y \notin I$ we know that y is contained in an edge E which contains a point not belonging to E_1. Consider $E_2 = \max \hat{\mathcal{E}}_y$. E_2 includes both $E_1 \backslash I$ and E. Since E_2 contains a point not belonging to E_1 it follows from the definition of I that $E_2 \in \mathcal{E}$. We conclude that $(X,\mathcal{E})$ contains two incomparable edges E_1 and E_2 with $E_1 \cap E_2 = E_1 \backslash I$.

CASE 2.1. $I = \{i\}$.

Suppose i is not a nest point. Then there are two incomparable edges F_1 and F_2 containing i. Choose $a_1 \in F_1 \backslash F_2$ and $a_2 \in F_2 \backslash F_1$. Then $(i, F_1, a_1, E_2, a_2, F_2, i)$ is a cycle of length three not containing an edge which contains all three vertices, contradicting the assumption that $(X,\mathcal{E})$ is super-balanced. Hence i is a nest point.

CASE 2.2. $|I| > 1$.

Let $i \in I$. Consider the subhypergraph of $(X,\mathcal{E})$ obtained by deleting $I\backslash\{i\}$ from X. By induction this hypergraph has two nest points; at least one of them, say y, is different from i. We have $y \notin E_1 \backslash (I\backslash\{i\})$ since all points from $E_1 \backslash (I\backslash\{i\})$ except i are contained in two incomparable edges, namely $E_1 \backslash (I\backslash\{i\})$ and E_2. By definition of I it follows that y is not contained in an edge of $\mathcal{E}$ which also contains a point of I, hence y is also a nest point of $(X,\mathcal{E})$. □

LEMMA 2.6. *Let* $(X,\mathcal{E})$ *be a super-balanced hypergraph satisfying Property 2.4 and let* y *be a nest point of the subhypergraph obtained from* $(X,\mathcal{E})$ *by deleting a vertex* x*, but not a nest point of* $(X,\mathcal{E})$*. Then* $\min \hat{\mathcal{E}}_y \notin \mathcal{E}$ *and* $\min \hat{\mathcal{E}}_y \cup \{x\} \in \mathcal{E}$*, where* $\hat{\mathcal{E}} = \mathcal{E}(\{x\})$.

PROOF. Since y is not a nest point of $(X,\mathcal{E})$ there are edges $F_1,F_2 \in \mathcal{E}$ such that $F_1\backslash\{x\} \subsetneq F_2\backslash\{x\}$ and $x \in F_1$, $x \notin F_2$. Since $\min \hat{\mathcal{E}}_y \subseteq (F_1 \cap F_2)$ it follows from Property 2.4 that $\min \hat{\mathcal{E}}_y \notin \mathcal{E}$ and hence $\min \hat{\mathcal{E}}_y \cup \{x\} \in \mathcal{E}$. □

PROOF OF THEOREM 2.2. Assume $(X,\mathcal{E})$ contains exactly one nest point x. Consider the subhypergraph obtained from $(X,\mathcal{E})$ by deleting x from X. By induction there are two nest points y_1 and y_2. Define $E_i = \min \hat{\mathcal{E}}_{y_i} \cup \{x\}$, where $\hat{\mathcal{E}} = \mathcal{E}(\{x\})$ $(i = 1,2)$. Since y_1 and y_2 are not nest points of $(X,\mathcal{E})$ it follows from Lemma 2.6 that $E_i \in \mathcal{E}$ $(i = 1,2)$. Since $x \in E_i$ $(i = 1,2)$ and x is a nest point it follows that E_1 and E_2 are comparable, say $E_1 \subseteq E_2$.

CASE 1. $E_1 = E_2$

Then y_1 and y_2 are contained in exactly the same edges. Consider the subhypergraph of $(X,\mathcal{E})$ obtained by deleting y_1 from X. It has two nest points z_1 and z_2. Clearly z_1 and z_2 are also nest points of $(X,\mathcal{E})$, contradicting our assumption that there is exactly one nest point of $(X,\mathcal{E})$.

CASE 2. $E_1 \subsetneq E_2$

Consider the partial hypergraph obtained from $(X,\mathcal{E})$ by deleting E_1 from $\mathcal{E}$. It has two nest points; at least one, say z, is different from x. If $z \notin E_1$, then z is also a nest point of $(X,\mathcal{E})$, contradicting the assumption that x was the only nest point of $(X,\mathcal{E})$. So assume $z \in E_1$. But then $\mathcal{E}_{y_2} \subset \hat{E}_z$, where $\hat{\mathcal{E}} = \mathcal{E}\backslash\{E_1\}$, and hence y_2 is also a nest point of $(X,\mathcal{E})$. Since $y_2 \neq x$ this again leads to a contradiction. □

We conclude that $(X,\mathcal{E})$ contains at least two nest points.

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