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AN INFINITE SERIES OF SYMMETRIC DESIGNS

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An infinite series of symmetric designs

by

A.E. Brouwer

ABSTRACT

We construct symmetric $2-(v,k,\lambda)$ designs where

$$v=2(q^h+q^{h-1}+\ldots+q)+1,$$
$$k=q^h$$

and

$$\lambda=\frac{1}{2}q^{h-1}(q-1)$$

whenever q is an odd primepower and $h \geqslant 1$.

KEY WORDS & PHRASES: *symmetric balanced incomplete block design*

Introduction

Many years ago R.M. WILSON produced a list of parameters for which symmetric block designs are known to exist. He writes "The phrase 'known to exist', means, more precisely, known to me.". No constructions were given, and later other authors (Beker & Piper [1], Mitchell [3], van Leyenhorst [2]) published their own version of such constructions. Looking at the parameter sets claimed by Wilson for which as far as I know no constructions appear in the literature I noticed that each of them had the form given in the abstract and indeed it turns out not to be too difficult to construct designs with these parameters. In the second part of this paper I reproduce Wilson's list, brought up to date by additions due to C.J. Mitchell and myself.

The construction

Let q be an odd prime power and let $h \geqslant 1$.
1. As first ingredient we need a $w \times w$ matrix B with entries in $\{0,\pm 1\}$ where $w=(q^h-1)/(q-1)$ such that $BB^t=q^{h-1}I$ and such that the matrix $(1-b_{ij}^2)_{ij}$ is the $0-1$ incidence matrix of points and hyperplanes in $PG(h-1,q)$. Now consider the matrix C of order q^h-1 with entries $c_{xy}=\chi(\text{trace } xy)$ where rows and columns are indexed by the elements of $GF(q^h)\setminus\{0\}$, trace is the trace map from $GF(q^h)$ to $GF(q)$ and $\chi: GF(q)\rightarrow\{0,\pm 1\}$ is the quadratic residue character (with $\chi^{-1}(0)=\{0\}$). Noticing that trace $ax=t$ defines (for fixed $a\neq 0$ and t) an affine hyperplane in the vector space $GF(q^h)$ over $GF(q)$ and that the hyperplanes trace $a_i x=t_i (i=1,2)$ are parallel if and only if $a_1/a_2 \in GF(q)$ one immediately sees that the rows of C indexed by x and x' are orthogonal iff $x/x' \notin GF(q)$.

It follows that if we index the rows and columns of B with representatives for the multiplicative cosets of $GF(q)\setminus\{0\}$ in $GF(q^h)\setminus\{0\}$ then B has all the required properties. (B is not uniquely defined: if not all the elements of $GF(q)$ are squares in $GF(q^h)$ then choosing another representative may multiply all elements in corresponding row or column by -1.)

2. Our second ingredient is a symmetric $2-(2q+1,q,\frac{1}{2}(q-1))$ design. Such a design may be obtained from a Hadamard matrix of order $2q+2$ by normalizing it in such a way that the first row and column contain $+1$ only, then deleting this row and column, and finally changing all -1's to 0's. (Hadamard matrices of order $2(q+1)$ do exist: for $q\equiv 3 \pmod 4$ by the Paley construction (followed by doubling) and for $q\equiv 1 \pmod 4$ by Theorem 8.39 in Wallis [ ].) Write its point-block incidence matrix in the following form:

	1	$2q$
q	$\underline{1}$	E
$q+1$	$\underline{0}$	A

Now A has row sums q and column sums $\frac{1}{2}(q+1)$ and E has row sums $q-1$ and column sums $\frac{1}{2}(q-1)$. Observe that replacing A by $J-A$ does not alter the inner products between columns.

3. Our third and last ingredient is the $q^h \times qw$ $0-1$ matrix G, the incidence matrix of points and hyperplanes in $AG(h,q)$.

Now define the following block matrix:

q^h	$\underline{j}$	E_0	E_1	....
		A_{00}	A_{01}	....
$(q+1)w$	$\underline{0}$	A_{10}	A_{11}	....
	$\vdots$			....
	1	$2qw$		

where $A_{ij} = \begin{cases} 0 \text{ if } b_{ij}=0 \\ A \text{ if } b_{ij}=1 \\ J-A \text{ if } b_{ij}=-1 \end{cases}$

and E_j is obtained from E by repeating each row q^{h-1} times, as follows: let the index j correspond to the $j-th$ parallel class of hyperplanes in $AG(h,q)$ (in some arbitrary numbering); now if $z \in AG(h,q)$, let row z of E_j be row k of E if z lies in the k-th hyperplane of this parallel class. (In other words, if G_j is the $q^h \times q$ submatrix of G corresponding to the j-th parallel class, then $E_j = G_j E$.) One checks easily that the $v \times v$ matrix thus obtained has row sums $k=q^h$ and the inner product of distinct columns is $\lambda = \frac{1}{2}q^{h-1}(q-1)$ so that we found the required symmetric design.

[Let us check the least trivial part of this statement and compute the inner product of two distinct columns of our matrix. If one of them is the first column everything is OK:

$$E_jJ = G_jEJ = \frac{1}{2}(q-1)G_jJ = \frac{1}{2}(q-1)q^{h-1}J = \lambda J.$$

If both of them belong to the same parallel class of $AG(h,q)$ then they lie in a column $(E_jA_{0j}.....)^{"t"}$ of our block matrix. Since the inner products of columns of A and those of $J-A$ are the same, we may assume that all A_{ij} are either A or 0. But in this case we just have the matrix $(EA)^{"t"}$ repeated q^{h-1} times so that the inner product of two columns again is $\frac{1}{2}(q-1)q^{h-1}$. Finally, if both columns belong to different parallel classes (say j and j') of $AG(h,q)$ then we find (since $E_j^tE_{j'} = E^tG_j^tG_{j'}E = q^{h-2}E^tJE = \frac{1}{4}(q-1)^2q^{h-2}$ and

$$\begin{bmatrix} A \\ A \end{bmatrix}^t \cdot \begin{bmatrix} A \\ J-A \end{bmatrix} = \frac{1}{2}(q+1)J)$$

inner product $\frac{1}{4}(q-1)^2q^{h-2} + \frac{1}{2}(q^{h-1}-q^{h-2}).\frac{1}{2}(q+1) = \frac{1}{2}(q-1)q^{h-1} = \lambda$.]

References

[1] H.J. Beker & F.C. Piper, *Some designs which admit strong tactical decompositions,* J. Combinatorial Theory 22 (1977) 38-42.

[2] D.C. van Leyenhorst, *manuscript.*

[3] C.J. Mitchell, *An infinite family of symmetric designs,* Discr.Math 26 (1979) 247-250.

MC, May 1982.

Appendix - The known symmetric designs
(essentially by R.M. Wilson)

A. Some constructions yielding symmetric designs with small index.

a. Mc Farland constructs designs with parameters

$$(v,k,\lambda) = (q^{d+1}.(\frac{q^{d+1}-1}{q-1}+1),\ q^d\frac{q^{d+1}-1}{q-1},\ q^d\frac{q^d-1}{q-1}),$$
$$n = q^{2d}$$

for prime powers q.
[R.L. Mc Farland, *A family of difference sets in non-cyclic groups,* J. Combinatorial Th. (A) 15 (1973) 1-10.]

b. From a regular Hadamard matrix of order $4t^2$ with constant diagonal one obtains a design with

$$(v,k,\lambda) = (4t^2, 2t^2-t, t^2-t),\quad n = t^2.$$

Such designs exist in particular when there is a Hadamard matrix of order $2t$ or when both $2t-1$ and $2t+1$ are prime powers. (cf. Wallis-Street-Wallis, Combinatorics: *Room squares, Sum - free sets, Hadamard matrices,* Lecture Notes in Math. 1292, Springer, Berlin etc. 1972, esp. pp. 341-346. Brouwer & van Lint, *Strongly regular graphs and partial geometries,* Section 8D (to appear in the procedings of the Silver Jubilee Conference, Waterloo 1982) .)

c. The adjacency matrix of the strongly regular graph with as vertices the elliptic or hyperbolic points off a quadric in $PG(2n,3)$ where two points are adjacent when they are joined by an elliptic line, is the incidence matrix of a symmetric design with parameters

$$(v,k,\lambda) = (\frac{1}{2}3^m(3^m \pm 1),\ \frac{1}{2}.3^{m-1}(3^m \mp 1),\ \frac{1}{2}.3^{m-1}(3^{m-1} \mp 1)),$$
$$n = 3^{2(m-1)}.$$

d. Van Leyenhorst constructs designs with parameters

$$(v,k,\lambda) = (u^3+u+1,\ u^2+1, u)$$

whenever both $u-1$ and u^2-u+1 are prime powers.

e. The family described in this note.

f. Mitchell constructs designs with parameters

$$(v,k,\lambda) = (q^{h+1}-q+1, q^h, q^{h-1})$$

for $h \geqslant 2, q > 2$ a prime power, whenever there exists an affine plane $AG(2,q-1)$. [C.J. Mitchell, *An infinite family of symmetric designs,* Discr. Math. 26 (1979) 247-250.]

g. Designs with a difference set listed in Baumert. [L.D. Baumert, Cyclic difference sets, Springer Lecture Notes in Math. 182, Berlin 1971] (See also: M. Hall, jr., Difference sets, in: Combinatorics (proceedings of the 1974 conference held at Nyenrode Castle, Breukelen), M. Hall, jr. & J.H. van Lint (eds.), Math. Centre Tracts 57 (1974) 1-26.)

h. Some special constructions.

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B. Table of admissible parameters

We list the triples $(v.k.\lambda)$ satisfying
(1) $(v-1)\lambda = k(k-1)$,
(2) the Bruck - Ryser - Chowla condition,
(3) $k^2-k+1>v>2k+1$
(thus eliminating the triples corresponding to finite projective planes and Hadamard designs), and
(4) $n: =k-\lambda \leqslant 30$.

n	v	k	λ	Solution	n	v	k	λ	Solution
4	16	6	2	*a*	15	61	25	10	*e*
6	25	9	3	*e,f*	16	154	18	2	?
7	37	9	2	*g*	16	115	19	3	?
7	31	10	3	*d*	16	96	20	4	*a*
9	56	11	2	*h*	16	85	21	5	*g*
9	45	12	3	*a*	16	78	22	6	?
9	40	13	4	*g*	16	70	24	8	?
9	36	15	6	*b*	16	66	26	10	*h*
10	41	16	6	*h*	16	64	28	12	*b*
11	79	13	2	*h*	18	191	20	2	?
11	49	16	5	*h*	18	79	27	9	*f*
12	71	15	3	*h*	19	211	21	2	?
12	61	16	4	*f*	19	155	22	3	?
13	81	16	3	?	19	101	25	6	*g*
13	69	17	4	*d*	19	85	28	9	?
14	121	16	2	?	20	139	24	4	?
15	71	21	6	?	20	121	25	5	?

n	v	k	λ	Solution	n	v	k	λ	Solution
21	131	26	5	?	25	100	45	20	*b*
21	109	28	7	*g*	26	127	36	10	?
21	85	36	15	?	27	407	29	2	?
22	201	25	3	?	27	291	30	3	?
22	127	28	6	?	27	177	33	6	?
22	97	33	11	?	27	141	36	9	?
23	301	25	2	?	27	121	40	13	*g*
23	103	34	11	?	27	111	45	18	?
25	352	27	2	?	28	311	31	3	?
25	253	28	3	?	28	249	32	4	?
25	204	29	4	?	28	171	35	7	?
25	175	30	5	*a*	28	149	37	9	?
25	156	31	6	*g*	28	131	40	12	?
25	133	33	8	*g*	28	113	49	21	*e*
25	120	35	10	?	29	265	33	4	?
25	112	37	12	?	29	181	36	7	?
25	105	40	15	?	30	239	35	5	?

The triples satisfying (1), (2), (3) with $30 < n \leqslant 100$ for which symmetric designs are known are:

n	v	k	λ	Solution	n	v	k	λ	Solution
31	223	37	6	*d*	64	256	120	56	*b*
36	176	50	14	*h*	66	265	121	55	*e*
36	144	66	30	*b*	72	721	81	9	*f*
37	197	49	12	*g*	73	739	82	9	*d*
45	181	81	36	*e*	75	311	125	50	*e*
48	253	64	16	*f*	81	891	90	9	*a*
49	441	56	7	*a*	81	820	91	10	*g*
49	400	57	8	*g*	81	378	117	36	*a*,*c*
49	196	91	42	*b*	81	364	121	40	$PG(5,3)$
54	241	81	27	*f*	81	351	126	45	*c*
56	505	64	8	*f*	81	324	153	72	*b*
64	640	72	8	*a*	91	365	169	78	*e*
64	585	73	9	*g*	100	621	125	25	*f*
64	341	85	21	*g*	100	400	190	90	*b*