A GENERALIZATION OF BARANYAI'S THEOREM
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A generalization of Baranyai's theorem

by

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ABSTRACT

The existence of resolvable parallelisms on a complete multipartite hypergraph is shown. As an application a question of P.J. Cameron is answered.

KEY WORDS & PHRASES: parallelism
1. INTRODUCTION

Let $X$ be a finite set which is the disjoint union of $r$ subsets $X_i$:

$$X = \bigcup_{j=1}^{r} X_j.$$  

Let $n = |X|$ and $n_j = |X_j|$ ($1 \leq j \leq r$). Let $N = \prod_{j=1}^{r} \{0, \ldots, n_j\}$ and define for $i \in N$:

$$\binom{n}{i} = \prod_{j=1}^{r} \binom{n_j}{i_j}$$

For each subset $A \subseteq X$ we define its characteristic as the rowvector $i_A = (|A \cap X_1|, \ldots, |A \cap X_r|) \in N$. Observe that $\binom{n}{i}$ is just the number of subsets of $X$ with characteristic $i$. Let a map $a : N \to \mathbb{N}$ be given. (We shall often write $a_i$ instead of $a(i)$.) A collection $C$ of subsets of $X$ is called an $(a)$-spread if

(i) for each $i \in N$ it contains exactly $a_i$ sets of characteristic $i$ and

(ii) each point of $X_j$ is contained in the same number $\lambda_j$ ($1 \leq j \leq r$) of elements of $C$.

If $\lambda = 1$ it is called an $(a)$-partition.

Observe that $\lambda$ is uniquely determined by the function $a$:

$$\sum_{i \in N} a_i \cdot i = \lambda \cdot n = (\lambda_1 n_1, \ldots, \lambda_r n_r).$$

We now have the following theorem:

**Theorem 1.** A collection of $\lambda$ $(a)$-spreads on $X$ such that each subset of $X$ with characteristic $i$ occurs exactly $a_i$ times among the members of the spreads exists if and only if

(i) for each $i \in N$ : $\lambda a_i = \binom{n}{i} a_i$,

(ii) $\sum_{i \in N} a_i \cdot i = n \cdot \lambda$

where $(\text{if } \lambda \neq 0)$ and the $a_i$ ($i \in N$) and $\lambda_j$ ($1 \leq j \leq r$) must be integers.
The stated conditions are obviously necessary: (i) counts the number of
sets with characteristic \( i \) in two ways, while (ii) counts in two ways the
number of times a point is covered. The sufficiency will be proved in the
next section.

Now consider some special cases:

First, if we set \( r = 1 \) and \( a_i = \delta_{ih} \) (then \( a_i = \delta_{ih} \cdot \frac{n_i}{h} \) and \( \lambda = \frac{1}{\lambda}(n-1) \))
we get the theorem of BARANYAI [1]:

**COROLLARY 1.1.** If \( h \mid n \) and \( \lambda \mid \frac{n-1}{\lambda} \) then the complete \( h \)-uniform hypergraph
on \( n \) vertices is \( \lambda \)-factorizable; in particular this is true for \( \lambda = \frac{h}{(n,h)} \).

Here a \( \lambda \)-factorization of a hypergraph \((X,E)\) is a partition of its edge-set
\( E = \bigcup_j E_j \) such that for each \( j \) and each \( x \in X \mid \{E_j \mid x \in E\} \mid = \lambda \) holds.

A \( \lambda \)-factorization is also called a *parallelism*.

The next special case, \( r = 2 \), will provide an answer to the question
of P.J. CAMERON [2]: For which \( h \) and \( n \) does there exist a parallelism on
the collection of all \( h \)-subsets of a given \( n \)-set \( X \) such that it induces a
parallelism on some \( \frac{1}{h} \)-subset \( X_1 \) of \( X \)?

That is, we would like to have a parallelism on \( X \) such that each parallel
class either contains only \( h \)-sets intersecting both \( X_1 \) and \( X_2 := X \setminus X_1 \) or
contains only \( h \)-sets entirely contained within \( X_1 \) or \( X_2 \). Clearly \( 2h \mid n \) is
necessary. Cameron knew of solutions for \( h = 2 \) or \( h = 3 \) and \( n = 12 \) or
\( n = 2h \), while J.-C. Bermond, J.I. Hall and the author constructed solutions
for \( h = 3 \) and \( 6 \mid n \) using resolvable triple systems.

But from the theorem above, taking \( r = 2 \), \( n_1 = n_2 = \frac{1}{h}n \), \( \lambda_1 = \lambda_2 = 1 \) and
some fixed \( g : a^g_{h-g} = a^g_{h-g} = 1 \) and all other \( a \)'s zero (so that
\( a^g_{h-g} = a^g_{h-g} \) if \( 2g \neq h \) and \( a^g_{h-g} = \frac{n}{h} \) if \( 2g = h \), while it is also
easy to check that \( \lambda \) is integral), it follows that there exists a parallelism on
all \( h \)-subsets intersecting \( X_1 \) in \( g \) or \( h-g \) points; now take the
union of these parallelisms for \( g = 0,1,\ldots,\lfloor \frac{1}{h} \rfloor \) to get the required sys-
tem:

**COROLLARY 1.2.** If \( 2h \mid n \) then there exists a parallelism on the collection of
all \( h \)-subsets of a given \( n \)-set which induces a parallelism on a \( \frac{1}{h} \)-subset.
Finally we mention a result announced in BARANYAI [1]:

Let $K^h_{r \times m}$ be the collection of all $h$-subsets $A \subset X$ such that

$$|A \cap X_j| \leq 1 \quad (1 \leq j \leq r),$$

where $|X_1| = \ldots = |X_r| = m$ (so that $n = rm$). Then

**COROLLARY 1.3.** Let $1 \leq h \leq r$ and $h | n \lambda$ and $\lambda | (r-1)!^m h^{-1}$. Then $K^h_{r \times m}$ is $\lambda$-factorizable.

**PROOF.** If $(\frac{r-1}{h-1})! \lambda m$ we can directly apply Theorem 1 to get a $\lambda$-factorization in which every $\lambda$-factor is an (a)-spread for the same function $a$. In the general case however, just as in the proof of the corollary 1.2, we need $\lambda$-factors of several types. The choice of the types can be done by an application of corollary 1.1 as follows: Let

$$\mu = \frac{h}{(h,r)}, \quad \text{and let } K^h_r = \bigcup_{j=1, \ldots, (r-1)!/(h-1)!/\mu} E_j,$$

be a $\mu$-factorization of the complete $h$-uniform hypergraph on $r$ vertices. Identifying sets $E \in E_j$ with 0-1 vectors of length $r$, we can consider each $E_j$ as a subset of $N$. Now apply Theorem 1 for each $j$ with $a_i = 1$ if $i \in E_j$ and $a_i = 0$ otherwise. (Then $\lambda = \frac{\mu}{\lambda} m^{-1}$ and $a_i = \frac{\lambda}{\mu} m$ (if $i \in E_j$) are integers.) This yields that for each $j$ the collection of subsets of $X$ with characteristic in $E_j$ is $\lambda$-factorizable, and hence $K^h_{r \times m}$ is $\lambda$-factorizable.

**PROOF OF THE THEOREM.** Let

$$X = \{x_1, \ldots, x_n\}, \quad \text{and } X_j = \{x_{m_j-1}^{j}, \ldots, x_{m_j}^{j}\},$$

where

$$m_s = \sum_{j \leq s} n_j.$$

We prove the theorem using induction with respect to $k$ and $s$, where $k$ ranges from 0 to $n$ and either $x_k^{s} \in X_s$ or $k = m_{s-1}$. The inductive assertion is:
Let $X(k) = \{x_1, \ldots, x_k\}$. There exists a collection of $k$ $\lambda$-factors $F(k)$ ($1 \leq g \leq l$) on the set $X(k)$, where each $F(k)$ is the disjoint union of sets $F(k)_{g,i} (i \in \mathbb{N})$ such that

1. $|F(k)_{g,i}| = a_i$ for $i \in \mathbb{N}$ and $1 \leq g \leq l$.
2. If $Y \in F(k)_{g,i}$ then for $j < s$ : $|Y \cap X_j| = i_j$.
3. If $Y \in X(k)$ then for each $i$ such that $Y \cap X_j = i_j$ for $j < s$, $Y$ occurs $a_i M_i \left(\frac{m_i - k}{i_i} |Y \cap X_s|\right)$ times in some $F(k)_{g,i}$, where

$$M_i = \binom{n_j}{i_j}.$$

The idea is that the $F(g)$ are the required $\lambda$-factors, and the $F(g)$ are the subsets of $F(g)$ consisting precisely of the sets with characteristic $i$.

The $F(k)$ and $F(k)_{g,i}$ will be their restrictions to $X(k)$, i.e. $F(k) = \{A \cap X(k) | A \in F(g)\}$ and for $F(k)_{g,i}$ likewise.

Note that $F(k)$ may contain the same set more than once, i.e. it is a selection rather than a set.

Given this interpretation, the conditions of the inductive hypothesis are clearly necessary, and it will appear below that they suffice.

Starting the induction with $k = 0$, $s = 1$, we are to construct collections $F(0)$ containing empty sets only, where the empty set occurs for each $i \in \mathbb{N}$ $a_i (\frac{m_i}{i_i})$ times in some $F(0)_{g,i}$, and $|F(0)_{g,i}| = a_i$. This is possible since by assumption $a_i$ and $a_i$ are integers and $a_i (\frac{m_i}{i_i}) = \lambda a_i$.

There are two kinds of induction step: steps that increment $k$ and steps that increment $s$ if $k = m$.

The latter are only a formality: suppose the induction hypothesis has been verified for $s = t$ and $k = m$, and let now $s = t + 1$.

2. requires that for $Y \in F(k)_{g,i}$ $|Y \cap X_t| = i_t$ but this follows from 3. since $\left(\frac{m_k - k}{i_t} |Y \cap X_t|\right)$ is nonzero only if $|Y \cap X_t| = i_t$. 

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3. requires that \( Y \) occurs \( \alpha_i \prod_{j=1}^{i} \binom{n_j}{i_j} \) times for such \( Y \), and this equals the hypothesis.

The former are implemented using a flow-through-network argument:
Suppose the collections \( F^{(k)}_g \) constructed for some \( k < m_s \). Then in order to get them for \( k+1 \) we have to choose \( \lambda_s \) sets from each collection \( F^{(k)}_g \) and adjoin the point \( x_{k-1} \) to them so that

\[
F^{(k+1)}_g = \{ Y \in F^{(k)}_g | Y \text{ not chosen} \} \cup \{ Y \cup \{ x_{k+1} \} | Y \text{ chosen} \}.
\]

Consider a directed network with vertices: source, sink, \( F^{(k)}_g \) \( (1 \leq g \leq s) \),
and edges from the source to each \( F^{(k)}_g \) from \( F^{(k)}_g \) to each \( F^{(k)}_g \) from \( F^{(k)}_g \) to \( Y \), iff \( Y \in F^{(k)}_g \), from \( Y \) to \( Y \) and from each \( Y \) to the sink.

A flow through this network is completely defined by its value on each of the edges \( (F^{(k)}_g, Y_i) \). Consider the flow with value \( i_s - |Y_nX_s| \) on each such edge. Through the vertex \( F^{(k)}_g \) the flow is

\[
\frac{1}{m_s - k} \sum_{Y \in F^{(k)}_g} \sum_{i \in N} (i_s - |Y_nX_s|) = \frac{\lambda_s}{m_s - k} (n_s - (k - m_s - 1)) = \lambda_s
\]

since \( \sum_{i \in N} \frac{1}{m_s - k} = \lambda_s \) and \( F^{(k)}_g \) restricted to \( X_s \cap X^{(k)} \) is a \( \lambda_s \)-factor.

Through the vertex \( i_s \), the flow is

\[
\frac{i_s - |Y_nX_s|}{m_s - k} \cdot \alpha_i M_i \binom{m_s - k - 1}{i_s - |Y_nX_s|} = \frac{1}{m_s - k} \binom{m_s - k - 1}{i_s - |Y_nX_s| - 1}
\]

which is an integer.

Now use the integrality theorem on flows in networks in the following form:

If there is a flow in a network with value \( \phi_i \) on edge \( e_i \), then there is a flow with value \( \psi_i \) on edge \( e_i \), where \( \phi_i - 1 < \psi_i < \phi_i + 1 \) and \( \psi_i \) is integral for each \( i \). [I.e. all flow values may be rounded either up or down in such a way that again a flow results. In particular if some flow value was integral it is not changed.]

(cf. Ford & Fulkerson [3], p. 19).
In this particular case the integrity theorem yields an integer flow through the network with flow \( \lambda_g \) through each vertex \( F^{(k)}_g \), i.e. the flow defines for each collection \( F^{(k)}_g \) \( \lambda_g \) elements \( Y \), each belonging to some known \( F^{(k)}_g \). Now if we adjoin the point \( x_{k+1} \) to these sets \( Y \) then, using that

\[
\begin{pmatrix}
\binom{m_s - k}{i_s - |Y_1 X_s|} \\
\binom{m_s - k - 1}{i_s - |Y_2 X_s|}
\end{pmatrix} = \binom{m_s - k - 1}{i_s - |Y_1 X_s|} + \binom{m_s - k - 1}{i_s - |Y_2 X_s|}
\]

it is readily verified that the new collections \( F^{(k+1)}_g \) and \( F^{(k+1)}_{g,i} \) satisfy the conditions 1,2 and 3.

This shows that the inductive hypothesis is true for \( k = n \) and \( s = r \), and therefore the theorem holds.

REFERENCES


