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A NEW 5-DESIGN

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by

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ABSTRACT

We construct some 4-(11,5,3) and 5-(12,6,3) designs without repeated blocks.

KEY WORDS & PHRASES: *t-design*

0. PRELIMINARIES

In DOYEN's table [2] the smallest unknown design is a 4-(11,5,3), or, in the other notation, a $S_3(4,5,11)$. Such a design cannot be obtained by taking 3 disjoint copies of S(4,5,11) since the maximum number of disjoint S(4,5,11)'s is two. KRAMER & MESNER [3] showed by computer search that a $S_3(4,5,11)$ cannot have a cyclic group of order 11 so that in particular its automorphism group is not transitive on the points. Let us compute some parameters of the design. First of all $\lambda_4 = 3$, $\lambda_3 = 12$, $\lambda_2 = 36$, $\lambda_1 = 90$, $b = \lambda_0 = 198$.

Next look at the number of times a given pattern occurs at given positions:

1	90	1111	3	11111	1
0	108	1110	9	11110	2
11	36	1100	15	11100	7
10	54	1000	15	11000	8
00	54	0000	9	10000	7
111	12			00000	2
110	24				
100	30				
000	24				

where in the case of patterns of size 5 we assumed that the positions of the patterns form a block of the design.

From b = 198 we infer that the automorphism group cannot contain elements of order 5 or 7. Indeed, if a 7-cycle acts then the four points not in the orbit of size 7 are either in 0 or in at least 7 blocks, contradicting $\lambda_4 = 3$. Also, if a 5-cycle acts and has two orbits of size 5 and one fixed point, then the number of blocks is congruent to 0,1 or 2 (mod 5), but $198 \equiv 3 \pmod 5$. Finally if a 5-cycle has 1 orbit of size 5 and 6 fixed points then at least one of the blocks covering four fixed points must intersect the orbit of size 5 and the fourtuple is covered at least five times. (The design we construct below has automorphism group S_4.)

We saw that given a block there are two others disjoint with it, and those two others intersect in 4 points:

11111 000000
00000 111110
00000 111101

hence we have a mapping ϕ: blocks → 4-tuples and since each 4-tuple is covered 3 times we have a mapping Ψ: blocks → blocks mapping each block into another intersecting it in one point:

11111	000000	B
00000	111110	
00000	111101	
00001	111100	$\Psi(B)$
.....	1111..	$\phi(B)$.

Suppose for some B_0 we have $\Psi^2(B_0) = B_0$. This means that the design contains six blocks looking like

11111000000	B_0
11110100000	B_1
11110010000	B_2
00001001111	$\Psi(B_0)$
00000101111	$\Psi(B_1)$
00000011111	$\Psi(B_2)$

From the nine blocks with pattern 0000...... , we know three already; the remaining six must intersect B_0, B_1 and B_2 so that they all look like

0000 111 (0011) B'

with two other zeros and two ones in the last four positions. But $\binom{4}{2} = 6$ hence all these blocks are in the design. Likewise we have all blocks

(0011) 111 0000 B".

The blocks disjoint from the B' are twelve blocks

(0111) 000 (0011)

where each pair on the right occurs exactly twice, and likewise we have 12 blocks

(0011) 000 (0111).

These are all 24 blocks with pattern .... 000 ...., but we have yet to decide in what way the two triples are choosen that go together with a pair on the other side. Unfortunately this is not determined uniquely, but after making a few reasonable assumptions one soon finds so many blocks that the remaining blocks of the design are determined uniquely.

1. THE DESIGN

Seeing that the resulting design induces a natural division of its point set into parts of size 4, 3 and 4 we asked our computer for all such designs invariant under some reasonable group. [The assumption of a non-trivial automorphism group seems essential - in fact the incentive for this reasearch was a talk by DRIESSEN reporting on an attempt to prove the nonexistence of the design, essentially by brute force, that had used already many hours of computer time.] To this end the pointset was chosen to be the set of expressions

$$V = \{a,b,c,d,ad+bc,ac+bd,ab+cd,bcd,acd,abd,abc\}.$$

With the symmetric group S_4 acting on the four letters (and hence on the points of the design) exactly one design was found (up to isomorphism). [The method used to obtain the design resembles the methods of KRAMER & MESNER; note however that using a set of expressions as pointset is more general than using the edges of a hypergraph - this method proves useful also in other contexts.]

We list its base blocks as characteristic funcitons, the set V being ordered as listed above. Each base block B is provided with a name and an orbit size, and we list the name of the base block of the orbit containing $\Psi(B)$.

name	block			orbitsize	$\Psi(B)$
0:	0000	001	1111	3	34
2:	0000	111	0011	6	15
4:	0001	001	0111	24	20
9:	0001	011	1001	12	27
10:	0001	011	1010	24	25
14:	0011	000	1101	12	25
15:	0011	001	0011	6	2
17:	0011	001	1100	6	26
20:	0011	010	0110	24	25
22:	0011	011	0001	24	10
25:	0011	110	0100	12	14
26:	0011	111	0000	6	17
27:	0111	000	0011	12	9
30:	0111	001	0100	12	14
31:	0111	001	1000	12	9
34:	1111	001	0000	3	0
				198	

One sees that

$$|\Psi^{-1}(B)| = \begin{cases} 0 & \text{for the blocks } 4,22,30,31 \\ 1 & \phantom{\text{for the blocks }} 0,2,10,15,17,20,26,27,34 \\ 2 & \phantom{\text{for the blocks }} 9,14 \\ 5 & \phantom{\text{for the blocks }} 25 \end{cases}$$

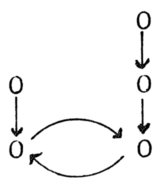
(Obviously $|\Psi^{-1}(B)|$ can never be larger than 5, since for each $p \in B$ there can be at most one block B_0 such that $\Psi(B_0) = B$ and $|B_0 \cap B| = \{p\}$.) Since we already know that the automorphism group does not contain elements of order 7 or 11 and moreover the orbit of block 25 is determined uniquely, S_4 is the full automorphism group of the design.

If we require somewhat less we get more: while in the case of S_4 we found 2 isomorphic solutions, in the case of imposed group A_4 we find 18 solutions (among which 2 nonisomorphic ones), and in the case of D_8 we find 32 solutions (among which 4 nonisomorphic ones).

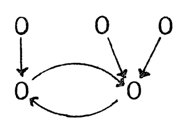
For the five designs found we list some characteristics connected with the Ψ-function:

Name	group	# of blocks B with $\|\Psi^{-1}(B)\| = j$ j = 0	1	2	3	4	5	cycle characteristics
A.	S_4 (A_4) (D_8)	72	90	24	--	--	12	15*[1 1] 12*[1 2] 12*[2 9]
B.	A_4	72	72	36	18	--	--	3*[1 1] 3*[1 1 1 3 1 1 1 3] 4*[3 2 1 7 3 2 1 7 3 2 1 7]
C.	D_8	64	106	4	12	12	--	7*[1 1] 4*[1 2] 4*[1 32] 2*[1 1 1 7 1 1 1 7]
D.	D_8	80	74	12	28	4	--	7*[1 1] 4*[2 15] 4*[7 16] 2*[1 1 1 3 1 1 1 3]
E.	D_8	72	82	16	28	--	--	7*[1 1] 4*[2 3] 4*[2 33] 2*[1 1 1 3 1 1 1 3] .

Here eg. 4* [2 3] means that in the directed graph on 198 vertices with the blocks of the design as verices and edge (B,C) if Ψ(B) = C there occur 4 cycles of length two such that its vertices are roots of trees of cardinality 2 and 3 respectively:



or



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2. EXTENSION TO A 5-DESIGN.

As shown by ALLTOP [1] any t-(2t+3,t+1,λ) design with t even can be extended to a (t+1)-(2t+4,t+2,λ) design by adding a point ∞ to the blocks of the given design and adjoining the complements of the old blocks. Hence from each 4-(11,5,3) we get a 5-(12,6,3) design.

In this way we get three 5-(12,6,3) designs:

α) a design of which all 12 derived 4-(11,5,3) designs are of type A

β) a design with all derived designs of type B.

γ) a design with 4 derived designs of types C, D and E each.

In all cases we only looked at the Ψ-characteristics of the produced designs but did not exhibit explicit isomorphisms between designs with the same Ψ-characteristics.

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