Itô's Formula

Exercise 26: Let B and $\bar{B}$ be two independent 1-dimensional Brownian motions. Show that

$$d(B\bar{B}) = Bd\bar{B} + \bar{B}dB.$$

Exercise 27: Show that the following stochastic processes are $\{\mathcal{F}_t\}$-martingales:

(a) $X_t = e^{\frac{1}{2}t} \cos B_t$

(b) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t)$

Exercise 28: Define the n-th Hermite polynomial, $n = 0, 1, \ldots$, by

$$H_n(t,x) \triangleq \frac{(-t)^n}{n!} e^{x^2/2t} \frac{d^n}{dx^n}\left(e^{-x^2/2t}\right)$$

Show that

$$\int_0^t H_n(s, B_s)dB_s = H_{n+1}(t, B_t), \quad n = 0, 1, \ldots;\ t \geq 0$$

or

$$dH_{n+1}(t, B_t) = H_n(t, B_t)dB_t.$$

($H_n(t, B_t)$ plays the role that $\frac{t^n}{n!}$ plays in ordinary calculus).

Hint:

$$\frac{d^n}{d\lambda^n}\left(e^{-\frac{(x-\lambda t)^2}{2t}}\right)\Big|_{\lambda=0} = (-t)^n \frac{d^n}{dx^n}(e^{-x^2/2t}).$$

$$\Rightarrow \quad \frac{d^n}{d\lambda^n}\left(e^{\lambda x - \frac{\lambda^2 t}{2}}\right)\Big|_{\lambda=0} = n!\, H_n(t,x).$$

$$\Rightarrow \quad e^{\lambda x - \frac{\lambda^2 t}{2}} = \sum_{n=0}^{\infty} \lambda^n H_n(t,x).$$