Solutions

Solution to 26: Set

$$X_t = \frac{B_t + \bar{B}_t}{\sqrt{2}}.$$

We claim that $\{X_t\}$ is a one dimensional Brownian motion. First note that $X_0 = 0$ a.s. and $\{X_t\}$ has independent increments. Next observe that since X is the sum of two independent $N(0, t/2)$ random variables, X_t is $N(0,t)$. A similar observation shows that $X_t - X_s$ is $N(0, t-s)$. This proves the claim. From 1-dimensional Itô's formula,

$$\begin{cases} d(X^2) & = 2XdX + dt \\ d(B^2) & = 2BdB + dt \\ d(\bar{B}^2) & = 2\bar{B}d\bar{B} + dt \end{cases}$$

Thus

$$\begin{aligned} d(B\bar{B}) &= d(X^2 - \frac{1}{2}B^2 - \frac{1}{2}\bar{B}^2) \\ &= 2XdX + dt - \frac{1}{2}(2BdB + dt) - \frac{1}{2}(2\bar{B}d\bar{B} + dt) \\ &= (B + \bar{B})(dB + d\bar{B}) - BdB - \bar{B}d\bar{B} \\ &= Bd\bar{B} + \bar{B}dB. \end{aligned}$$

Solution to 27:

(a)

$$\begin{aligned} d(e^{\frac{1}{2}t}\cos B_t) &= \frac{1}{2}e^{\frac{1}{2}t}\cos B_t dt - \frac{1}{2}e^{\frac{1}{2}t}\sin B_t dB_t - \frac{1}{2}e^{\frac{1}{2}t}\cos B_t dt \\ &= -e^{\frac{1}{2}t}\sin B_t dB_t \end{aligned}$$

Since $-(e^{\frac{1}{2}t}\sin B_t) \in \mathcal{H}^2$ result follows.

(b)

$$\begin{aligned} d((B_t + t)\exp(-B_t - \frac{1}{2}t)) &= (B_t + t)\exp(-B_t - \frac{1}{2}t)(-1)dB_t \\ &\quad + \exp(-B_t - \frac{1}{2}t)(dB_t + dt) + \exp(-B_t - \frac{1}{2}t)(-1)dt \\ &= \exp(-B_t - \frac{1}{2}t)(1 - t - B_t)dB_t \end{aligned}$$

Since $\exp(-B_t - \frac{1}{2}t)(1 - t - B_t) \in \mathcal{H}^2$ result follows.

Solution to 28: Using the limit

$$e^{\lambda x - \lambda^2 t/2} = \sum_{n=0}^{\infty} \lambda^n H_n(t, x)$$

$$M_t = e^{\lambda B_t - \lambda^2 t/2} = \sum_{n=0}^{\infty} \lambda^n H_n(t, B_t)$$

By Itô's formula,

$$\begin{cases} dM_t = \lambda M_t dB_t \\ M_0 = 1 \end{cases}$$

$$\Rightarrow \quad M_t = 1 + \lambda \int_0^t M_s dB_s; \quad t \geq 0.$$

Plugging in above expression for M_t

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n H_n(t, B_t) &= 1 + \lambda \int_0^t \sum_{n=0}^{\infty} \lambda^n H_n(s, B_s) dB_s \\ &= 1 + \sum_{n=1}^{\infty} \lambda^n \int_0^t H_{n-1}(s, B_s) dB_s \end{aligned}$$

Identity holds for all λ, equate coefficients on both sides of λ^n and the result follows.