Itô's Formula

Exercise 29: Suppose that $\{X_t\,;\,t\geq 0\}$ is a standard vector process described by

$$\vec{X}_t=\int_0^t A(s)\vec{X}_s ds+\int_0^t G(s)d\vec{B}_s \tag{1}$$

where $\{\vec{B}_t\,;\,t\geq 0\}$ is a d-dimensional Brownian motion, $G(t)$ is a continuous $n\times d$ matrix-valued function and $A(t)$ a continuous $n\times n$ matrix-valued function. Suppose that $\Phi(t,s)$ is the state transition matrix for the vector differential equation

$$\dot{\vec{x}}(t)=A(t)\vec{x}(t)$$

which means that $\Phi(t,s)$ is the $n\times n$ matrix-valued function that satisfies

$$\frac{\partial\Phi(t,s)}{\partial t}=A(t)\Phi(t,s)$$
$$\Phi(s,s)=I.$$

Furthermore, for $t>s>\tau$,

$$\Phi(t,s)\Phi(s,\tau)=\Phi(t,\tau).$$

Show that equation (1) has the unique solution

$$\vec{X}_t=\int_0^t\Phi(t,s)G(s)d\vec{B}_s.$$

Exercise 30: Solve the stochastic differential equation

$$dX_t=rdt+\alpha X_t dB_t$$

where r and α are real numbers.

Hint: Multiply by the "integrating factor" $F_t=\exp(-\alpha B_t+\frac{1}{2}\alpha^2 t)$.

Exercise 31: The geometric mean reversion process X_t is defined as the solution of the stochastic differential equation

$$dX_t=\kappa(\alpha-\ln X_t)X_t dt+\sigma X_t dB_t\,;\quad X_0=x>0$$

where κ, α, σ and x are positive constants.

(a) Show that the solution of this equation is given by

$$X_t=\exp\left(e^{-\kappa t}\ln x+(\alpha-\frac{\sigma^2}{2\kappa})(1-e^{-\kappa t})+\sigma e^{-\kappa t}\int_0^t e^{\kappa s}dB_s\right).$$

Hint: Use the substitution $Y_t=\ln X_t$.

(b) Show that

$$\mathbb{E}[X_t]=\exp\left(e^{-\kappa t}\ln x+(\alpha-\frac{\sigma^2}{2\kappa})(1-e^{-\kappa t})+\frac{\sigma^2(1-e^{-2\kappa t})}{2\kappa}\right).$$