Solutions

Solution to 29:

$$\begin{aligned}\int_0^t A(s)X_s ds &= \int_0^t A(s)\left[\int_0^s \Phi(s,\tau)G(\tau)dB_\tau\right]ds\\ &= \int_0^t\int_0^s A(s)\Phi(s,\tau)G(\tau)dB_\tau ds\\ &= \int_0^t \left(\int_\tau^t A(s)\Phi(s,\tau)ds\right)G(\tau)dB_\tau\end{aligned}$$

$$\begin{aligned}\int_0^t A(s)X_s ds + \int_0^t G(s)dB_s &= \int_0^t \left(I + \int_\tau^t A(s)\Phi(s,\tau)ds\right)G(\tau)dB_\tau\\ &= \int_0^t \Phi(t,\tau)G(\tau)dB_\tau = X_t\end{aligned}$$

because

$$\Phi(t,\tau) = I + \int_\tau^t A(s)\Phi(s,\tau)ds.$$

Solution to 30: By Itô formula,

$$\begin{aligned}dF_t &= F_t(-\alpha dB_t + \frac{1}{2}\alpha^2 dt) + \frac{1}{2}F_t\alpha^2 dt\\ &= F_t(-\alpha dB_t + \alpha^2 dt)\end{aligned}$$

$$\begin{aligned}d(F_tX_t) &= F_t dX_t + X_t dF_t + dF_t dX_t\\ &= F_t dX_t + X_t F_t(-\alpha dB_t + \alpha^2 dt) + (-\alpha F_t dB_t)(\alpha X_t dB_t)\\ &= F_t(dX_t - \alpha X_t dB_t) = F_t r dt\end{aligned}$$

$$\begin{aligned}F_tX_t &= F_0X_0 + \int_0^t rF_s ds\\ X_t &= X_0F_t^{-1} + F_t^{-1}\int_0^t rF_s ds\\ &= X_0\exp(\alpha B_t - \frac{1}{2}\alpha^2 t) + r\int_0^t \exp(\alpha(B_t - B_s) - \frac{1}{2}\alpha^2(t-s))ds\end{aligned}$$

Solution to 31:

(a) $Y_t = \ln X_t$

$$\begin{aligned}dY_t &= \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}\sigma^2X_t^2dt\\ &= \frac{1}{X_t}[k(\alpha - Y_t)X_t dt + \sigma X_t dB_t] - \frac{1}{2}\sigma^2 dt\\ &= -kY_t dt + k(\alpha - \frac{\sigma^2}{2k})dt + \sigma dB_t\end{aligned}$$

This is Ornstein Uhlenbeck process with extra constant term.

(b) X_t is lognormal. The result follows from standard formula for expectation of a lognormal random variable.