TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science

MASTER'S THESIS

Graphs related to $E_7(q)$

A quest for distance-transitivity

by

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Abstract

Distance-transitive graphs stand out from the crowd by their high degree of symmetry. Thanks to this symmetry such graphs can be described very efficiently. A distance-transitive graph is determined uniquely by its automorphism group, a corresponding subgroup that stabilizes a vertex in the graph, and an arbitrary edge. The question arises for which simple groups and corresponding subgroups, there exists a distance-transitive graph in which these groups play the roles as described above. Using the classification of all finite simple groups, many researchers have tried to answer this question. Much work has already been done. There exists, however, a small list with *hard* cases, that still need to be investigated.

In this thesis we examine one of these cases. We create a general graph structure with automorphism group $E_7(q)$ and vertex-stabilizer $A_7(q) \cdot 2$ over a field of characteristic 2. Using character theory, we show that such a graph cannot be distance-transitive. In this manner this thesis contributes to the classification of all primitive distance-transitive graphs.

Abstract

Afstands-transitieve grafen vallen op door hun hoge graad van symmetrie. Dankzij deze symmetrie is het mogelijk om deze grafen op een zeer efficiënte manier te beschrijven. Een afstands-transitieve graaf wordt uniek vastgelegd door middel van zijn automorfismengroep, een bijbehorende ondergroep die een punt van de graaf stabiliseert, en een willekeurige kant in de graaf. Dit roept de vraag op, voor welke combinaties van een enkelvoudige groep en een ondergroep, er een afstands-transitieve graaf bestaat, waarop deze groepen werken zoals hierboven is beschreven. Door gebruik te maken van de classificatie van de eindige enkelvoudige groepen wordt er al lang hard gezocht naar een antwoord op deze vraag. Er is al veel werk verzet, maar er zijn nog enkele *moeilijke* gevallen over.

In deze thesis construeren we een algemene graaf met automorfismengroep $E_7(q)$ en puntstabilisator $A_7(q) \cdot 2$ over een lichaam met karakteristiek 2. Met behulp van karakter-theorie tonen we aan dat een dergelijke graaf niet afstands-transitief kan zijn. Hierdoor draagt deze thesis bij aan de classificatie van alle primitieve afstands-transitieve grafen.

Preface

This Masters thesis is the result of nine months of research at the department of Discrete Algebra and Geometry of the Eindhoven University of Technology (TU/e). With this thesis, and the research it is based on, I conclude my study of Industrial and Applied Mathematics and obtain the degree Master of Science. My research is part of an ongoing effort to classify all primitive distance-transitive graphs. I was introduced to this area by prof. dr. Arjeh M. Cohen, who suggested and supervised this project.

Writing this thesis was not a one man show; I would like to take a moment to express my thanks for the help and support I received. First of all I would like to thank Arjeh Cohen. He supervised my work on this project, but did much more than that. He always had the time and patience to help me out when I was stuck. He was always full of enthusiasm and - which is just as important - managed to pass it on to me. On the multiple occasions that I was having doubts on whether my project would come to a good end, talking to him was very encouraging and stimulating. I also want to thank dr. Scott H. Murray; not only did he help me with all the MAGMA problems I encountered, but he was also always available to discuss any other problem I was working on. Furthermore I want to thank the extra members of my exam committee, prof. dr. A.E. Brouwer, dr. F.G.M.T.Cuijpers and dr. R.A. Pendavingh for taking place in this committee and for working their way through this thesis.

There is also a number of people who may not directly have influenced the contents of this thesis, but whose presence was invaluable nevertheless; they made the TU/e a nice place to work. Thanks to all the members of the Discrete Mathematics group and to my friends in the carpeted hallway, for the many lunch and tea breaks we had. I will miss them! A special thanks goes out to all of my friends, my family and to Esther in particular for their love and support. I hope that for the next few months this thesis will get a nice place on their night tables!

Contents

1 Introduction

People have always been fascinated by symmetry. Symmetry plays a role in nature, in art, and also in mathematics. Famous historical examples of symmetrical mathematical structures are the great pyramids and the Platonic solids. These structures can be represented by graphs. Although all of the mentioned examples have a high degree of symmetry, some of them are more symmetric than others. The Platonic solids belong to a class of graphs that we call *distance-transitive graphs.* They are considered to be the most symmetric (non-trivial) graphs around.

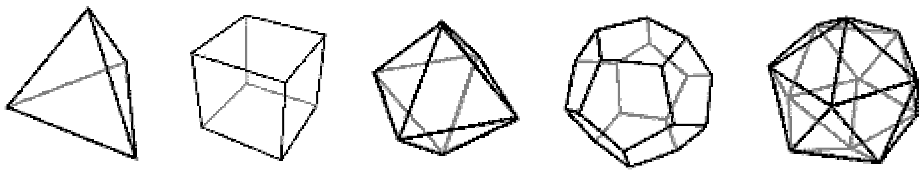


Figure 1: The Platonic solids

A graph $\Gamma = (V, E)$ is called distance-transitive (see [3], [5]) if for every two pairs of vertices (v, w) and (x, y) of the graph, such that v and w are at distance i of each other and vertices x and y are at distance i from each other, there exists a symmetry of the graph that takes (v, w) to (x, y). More formally we say that the *automorphism group* of Γ acts transitively on all distance sets of $V \times V$. A nice property of these distance-transitive graphs is that there is an efficient way to describe them. The stabilizer of an arbitrary vertex v in the graph Γ is a subgroup G_v of the automorphism group $G := Aut(\Gamma)$. It acts transitively on all sets of vertices at a fixed distance of v. Hence the groups G, G_v, and a group element $g \in G$ such that v and v^g are adjacent, are enough to describe Γ.

A result by Cameron, Praeger, Saxl, and Seitz [6] tells us that the number of distance-transitive graphs of any given degree $d > 2$ is finite. Since there are only finitely many of them, for any given degree, it is a logical step to aim for a classification of all of them. Indeed, a project was started to classify all distance-transitive graphs (see the survey by A. M. Cohen [10])[1]. Thanks to a result by D. H. Smith [28], we can interpret the *primitive* distance-transitive graphs as the building blocks for all distance-transitive graphs, hence they are a good place to start this classification. Once all the primitive distance-transitive graphs are classified, there are two techniques that can be used to construct new distance-transitive graphs from these building blocks, namely *bipartite doubling* and *antipodal covering.* Praeger, Saxl, and Yokoyama [25] showed that the automorphism group of a primitive distance-transitive graph either acts distance-transitively on a Hamming graph, it is affine, or it is almost simple.

Thanks to the classification of all finite simple groups, there is a finite list of groups and corresponding maximal subgroups that might give rise to a primitive distance-transitive graph (see [11]). For each of these cases it needs to be checked whether there exists a corresponding distance-transitive graph. A lot of work has already been done (see [10]). The remaining cases are mostly found among the exceptional Lie groups. In this thesis we investigate the open case where the automorphism group is the group $E_7(q)$. We give an overview of all its possible maximal subgroups that might give rise to distance-transitive graph. Focussing on the subgroup $A_7(q) \cdot 2$ we construct a general coset graph and show that this graph cannot be distance-transitive. This is a direct result from the main theorem of this thesis:

[1] Note that already before this project started, N. Biggs and D. H. Smith determined all trivalent distance-transitive graphs. This result was extended to the classification of all distance-transitive graphs with valency ≤ 13, mostly thanks to work by D. H. Smith [29] (for valency 4), A. A. Ivanov, A. V. Ivanov, and I. Faradjev [15], [19]

Theorem 5.3 *Let G be the Chevalley group of Lie type E_7 and of adjoint isogeny type over a field* $\mathrm{GF}(q)$ *of characteristic 2. Let H be the subgroup of G of Lie type A_7. Finally, let $-x_{\alpha_0}$ be the highest root of the Lie algebra L_{E_7} of adjoint isogeny type and define $Y := \{ \mathbb{F}_q x_{\alpha_0}^g \mid g \in G \}$. Then there are exactly 7 H-orbits on Y.*

In Section 2 we introduce some basic concepts and notations that are necessary for the understanding of this thesis. This introduction will cover concepts from graph theory (2.1), Lie theory (2.2), and character theory (2.3). Those familiar with these areas of mathematics can freely skip this section and use it only as a reference.

In Section 3 we describe a distance-transitive graph by constructing a coset graph of its automorphism group and a maximal subgroup that stabilizes a vertex of the original graph. We explain how this connection between graph theory and group theory makes it possible to classify all primitive distance-transitive graphs.

Among the exceptional Lie groups there are still groups of which it is unknown if they can be interpreted as the automorphism group of a distance-transitive graphs. One of these groups for which there are still some open cases is $E_7(q)$. In Section 4 we discuss these open cases. We focus on the maximal subgroup $A_7(q) \cdot 2$ of $E_7(q)$ and construct the coset graphs that can be obtained from these groups. In Section 5 we state the main theorem of this thesis; this graph cannot be distance-transitive. The proof relies on the fact that there are more than 5 different $A_7(q) \cdot 2$ orbits on the root elements of L_{E_7}. This connection is explained. The rest of this thesis describes the hunt for all 7 of these orbits, which can be divided in a number of steps.

In Sections 6 and 7 we set up representations for all groups and Lie algebras involved, which is convenient for computations. Sections 8 and 9 contain some more preparational work; some theory is developed to determine the stabilizers of root elements.

In Section 10 we determine the stabilizers for representatives from each of the 7 $A_7(q) \cdot 2$ orbits on the (conjugates of) root elements of L_{E_7}. The results of Section 10 are summarized in Section 11 and used to compute the orbit sizes. From the results we conclude that the 7 orbits that we found are indeed all orbits. In this manner, we finish the proof of Theorem 5.3 in Section 5. Furthermore, this section contains some recommendations for further research.

This thesis contains 3 appendices. Appendix A contains some more detailed explanation of theory used in this thesis, that goes beyond the scope of Section 2. In the other 2 appendices we deal with subjects related to the research, but not necessary for the understanding of the main results. In Appendix B we discuss some other methods that are widely used to determine if a certain graph is distance-transitive. Finally, Appendix C contains some comments on the benefits and disadvantages of using Magma on this project.

2 Preliminaries

In this section some basic theory of graphs, some Lie theory, and some character theory is developed. We do not intend to give a complete treatment of these areas, but merely wish to introduce some definitions and some theory, used in this thesis. We omit all proofs, but do refer to literature where needed. In section 2.1 we introduce the concept of a graph. In particular we introduce distance-transitive graphs. Section 2.2 deals with Lie groups and Lie algebras. In section 2.3 we introduce some basic concepts from character theory. The reader who is familiar with these topics can freely skip this section entirely, or use it for reference while reading the main part of this thesis.

2.1 An introduction to graphs

This section contains an introduction to graphs, and in particular, distance-transitive graphs. A more elaborate introduction into algebraic graph theory can be found in for example [3] or [17]. A very nice and readable introduction to distance-transitive graphs can be found in [2].

Definition 2.1 *A graph* $\Gamma = (V, E)$ *consists of a set of vertices* $V = V(\Gamma)$ *and a set of edges* $E = E(\Gamma)$*. An edge is a pair of vertices from* V*.*

We can represent a graph graphically by representing the vertices by dots, and drawing an arrow from dot A to dot B whenever the pair $\{A, B\}$ is an edge in E. We are only interested in a subclass of graphs, namely in those graphs that

- are undirected, i.e., each edge is an unordered pair of vertices and therefore represented by a line instead of an arrow;
- are connected, i.e., every two vertices of the graph are connected through a series of edges. Such a series of edges is called a *path*;
- have no loops, i.e., there is no edge from a vertex to itself;
- have at most one edge in between of each pair of vertices;
- have a finite number of vertices.

Whenever we talk about a graph in this thesis, we mean a graph that satisfies these properties.

Let $\Gamma = (V, E)$ be a graph. The number of edges incident to a vertex is called its *valency* or *degree*. The maximum of all valencies in the graph is called the *degree* of the graph. Two vertices $v, w \in V$ are called *adjacent* if $\{v, w\}$ is an edge. We also denote adjacency between v and w by $v \sim w$. The distance $\mathrm{d}(v, w)$ between v and w is the length of the shortest path between them. This function is well defined since we assume Γ to be connected, and it induces a partition of the set $V \times V$ into distance sets

$$D_i(\Gamma) := \{\, (v, w) \in V \times V \mid \mathrm{d}(v, w) = i \,\}.$$

Note that $D_1(\Gamma)$ is simply the edge set as introduced before. If we fix a vertex $v \in V$ then, we can define a partition of V into distance sets by

$$D_i(v) := \{\, w \in V \mid \mathrm{d}(v, w) = i \,\}.$$

The *diameter* of a graph is the maximum distance between two vertices.

Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are equal if $V_1 = V_2$ and $E_1 = E_2$. They are called *isomorphic* if there exists some bijection $\phi\colon V_1 \to V_2$ such that $\phi(v_1) \sim \phi(v_2)$ if and only if $v_1 \sim v_2$. The bijection ϕ is called an isomorphism. An isomorphism from a graph Γ to itself is called an *automorphism* of Γ. Clearly the set of all automorphisms of Γ form a group. We denote it with $\mathrm{Aut}(\Gamma)$ and it is called the *automorphism group* of Γ. Note that the automorphism group of Γ can be seen as a *permutation group* on the set of vertices V of Γ.

Let $G := \mathrm{Aut}(\Gamma)$ be the automorphism group of the graph Γ. We denote the image of a vertex v under the action of an element $g \in G$ by v^g. By v^G we denote the G-orbit of v. By G_v we denote the set of group elements that stabilize v, thus $G_v := \{\, g \in G \mid v^g = v \,\}$. The following lemma, which is known as the Orbit-Stabilizer-Lemma, will prove to be useful.

Lemma 2.1 *Let* G *be a permutation group acting on* V *and let* v *be a point in* V*. Then*

$$|G_v||v^G| = |G|.$$

The group G is called *vertex-transitive* if for each $v, w \in V$, there exists an element $g \in G$ such that $v^g = w$. It is called *edge-transitive* if for all $\{v, w\}, \{x, y\} \in E$ there exists a $g \in G$ such that $v^g = x$ and $w^g = y$. Finally we get to the definition of a distance-transitive group and graph.

Definition 2.2 *Let* $\Gamma = (V, E)$ *be a graph and let* G *be a permutation group acting on* V*. Then* G *is called* distance-transitive *if it acts transitively on all distance sets of* Γ*, i.e. if for all* $v, w, x, y \in V$ *such that* $\mathrm{d}(v, w) = \mathrm{d}(x, y) = i$ *for some* $i \in \{0, \ldots, d\}$ *there exists a* $g \in G$ *such that* $v^g = x$ *and* $w^g = y$*. The graph* Γ *is called a* distance-transitive graph (or DTG) *if its automorphism group acts distance-transitively on it.*

For an arbitrary group G that acts on a vertex set V of a graph Γ, there is also a natural action of G on the set $V \times V$. The orbits of G on $V \times V$ are called *orbitals* (of G). An orbital $\mathcal{O}$ is called *self-paired* if for all $v, w \in V$ we have that $(v, w) \in \mathcal{O} \Leftrightarrow (w, v) \in \mathcal{O}$. The number of G-orbits on $V \times V$ is called the *permutation rank* of G on V. If G is a distance-transitive group, then the permutation rank is equal to the number of distance sets of Γ. Thus if Γ has diameter d, the permutation rank is equal to $d + 1$.

Lemma 2.2 *A graph Γ with diameter d and automorphism group G is distance-transitive if and only if it is vertex-transitive and if G_v is transitive on the set $D_i(v)$ for each $i = 0, \ldots, d$ and for all $v \in V$.*

There are two special classes of graphs we introduce.

Definition 2.3 *A graph $\Gamma = (V, E)$ is called* bipartite *if there exists a partition $V = V_1 \cup V_2$ such that for every edge $\{v, w\} \in E$, either $v \in V_1$ and $w \in V_2$ or $w \in V_1$ and $v \in V_2$. This means that there exists a colouring of the vertices using two colours, such that no two adjacent vertices have the same colour.*

Definition 2.4 *A graph $\Gamma = (V, E)$ of diameter d is called* antipodal *if for all $v \in V$ and all $u, w \in D_d(v)$, either $u = w$ or $\mathrm{d}(u, w) = d$.*

2.2 An introduction to Lie algebras and Lie groups

In this section we borrow freely from Carter [7]. Other nice introductions to Lie algebras are [21] and [32]. For a nice glossary of the terminology of Lie algebras, see [16].

2.2.1 Roots

Let V be some finite dimensional Euclidean space, equipped with an inner product $\langle \cdot, \cdot \rangle$. For each non-zero vector v in V the reflection of some vector u in the hyperplane orthogonal to v is given by

$$w_v(u) := u - \frac{2\langle v, u \rangle}{\langle v, v \rangle} v.$$

Definition 2.5 *A subset Φ of V is called a* root system *(of V) if*

- *Φ is a finite set of non-zero vectors;*
- *Φ spans V;*
- *for every $\alpha, \beta \in \Phi$, $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ is a rational integer;*
- *for every $\alpha, \beta \in \Phi$, also $w_\alpha(\beta) \in \Phi$;*
- *for every $\alpha \in \Phi$ it holds that $\lambda\alpha \in \Phi$ if and only if $\lambda = \pm 1$.*

The elements of Φ are called *roots*. We usually denote them with the Greek letters α, β.

Definition 2.6 *A subset Π of Φ is called a* fundamental system *if*

- *Π is linearly independent;*
- *each element from Φ can be expressed as an integral sum of elements from Π with either all non-positive coefficients or all non-negative coefficients.*

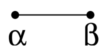
Let Φ^+ denote the set of all *positive* roots in Φ; these are the roots that can be expressed as a positive linear combination of fundamental roots. In a similar way Φ^- denotes the set of all *negative* roots.

We can describe the positions of a set of fundamental roots with respect to each other by means of a *Dynkin diagram.* This is a graph in which the nodes represent the fundamental roots and in which adjacency between two nodes depends on the position of the corresponding roots. The nodes with the labels α and β are

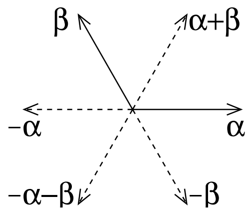
- not connected if the vectors α and β are perpendicular;
- connected by a single line if $\angle(\alpha, \beta) = 2\pi/3$;
- connected by a double line if $\angle(\alpha, \beta) = 3\pi/4$;
- connected by a triple line if $\angle(\alpha, \beta) = 5\pi/6$.

All simple Lie algebras (see section 2.2.2) have a fundamental system that can be described by a connected Dynkin diagram as defined above. In particular, the angles mentioned above are the only possible angles that can occur between two fundamental roots of a simple Lie algebra.

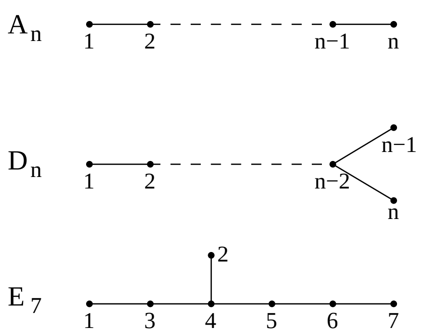
Example:
Let $\Phi \subset \mathbb{R}^2$ be a root system with fundamental roots α, β, which are inclined at $2\pi/3$. Its Dynkin diagram is given by



The corresponding root system Φ consists of the roots $\alpha, \beta, \alpha+\beta, -\alpha, -\beta, -\alpha-\beta$ and their positions with respect to each other are shown in the figure below.



In this thesis 3 different types of root systems will play a role. Here we give the Dynkin diagrams of the types A_n, $(n \geq 1)$, D_n, $(n \geq 4)$, E_7. For more information on each of these root systems and their corresponding Lie algebras and Lie groups, see for example [7].



The nodes are labeled by the indices of the fundamental roots in the corresponding root systems (assuming that enumeration of the roots starts with the fundamental roots) and follow Bourbaki's convention [4].

We next introduce the concept of a *root datum*, which roughly consists of a pair of root systems in duality. The following definition is from [12].

Definition 2.7 *Let X and Y be free $\mathbb{Z}$-modules of finite rank such that there exists a bilinear pairing $\langle \cdot , \cdot \rangle_\sharp \colon X \times Y \to \mathbb{Z}$ putting them in duality. Now let Φ be a root system of X and let Φ^* be the dual root system of Y. For each $\alpha \in \Phi$ we denote its dual by α^*. We call $\mathcal{R} = (X, \Phi, Y, \Phi^*)$ a* root datum *if for all $\alpha \in \Phi$*

- *the linear maps $w_\alpha \colon X \to X$ given by $x \mapsto x - \langle x, \alpha^* \rangle_\sharp \alpha$ and $w_{\alpha^*} \colon Y \to Y$ given by $y \mapsto y - \langle \alpha, y \rangle_\sharp \alpha^*$ are both reflections;*
- *Φ is closed under the action of w_α;*
- *Φ^* is closed under the action of w_{α^*}.*

The elements of Φ^* are called *coroots*. Note that if $\mathcal{R} = (X, \Phi, X, \Phi^*)$ and we define the coroots α^* by $\alpha^* := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ for each $\alpha \in \Phi$, then $\langle \alpha, \beta^* \rangle_\sharp = \frac{2\langle \alpha, \beta^* \rangle}{\langle \alpha, \alpha \rangle}$ just as we saw in the beginning of this section.

The group generated by all reflections w_α, for $\alpha \in \Phi$ is called the *Weyl group* of Φ and denoted by $W(\Phi)$ or simply W.

Lemma 2.3 *The Weyl group $W(\Phi)$ is generated by the fundamental reflections $w_\alpha, \alpha \in \Pi$. Moreover, all roots $\beta \in \Phi$ can be obtained from a fundamental root by applying a finite number of fundamental reflections.*

Lemma 2.4 *Each Weyl group contains a unique* longest element*, which is an involution that maps all positive roots to negative roots. It is usually denoted by w_0.*

2.2.2 Lie algebras

Definition 2.8 *A* Lie algebra *is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$, that is equipped with a product $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the* Lie bracket*, such that*

1. *$[x, x] = 0$ for all $x \in \mathfrak{g}$;*
2. *$[x, y]$ is bilinear for all $x, y \in \mathfrak{g}$;*
3. *$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$.*

The Lie multiplication is easily seen to be anti-commutative. Given some associative algebra A where the product of $x, y \in A$ is given by xy, one can turn it into a Lie algebra by equipping it with the Lie bracket $[x, y] = xy - yx$. For $\mathfrak{g}$ a Lie algebra and $\mathfrak{g}_1, \mathfrak{g}_2$ two arbitrary subspaces of $\mathfrak{g}$, we define $[\mathfrak{g}_1, \mathfrak{g}_2] := \{ [x, y] \mid x \in \mathfrak{g}_1, y \in \mathfrak{g}_2 \}$. Now $\mathfrak{g}_1$ is called a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$. It is called an ideal of $\mathfrak{g}$ if $[\mathfrak{g}_1, \mathfrak{g}] \subseteq \mathfrak{g}_1$. We call a Lie algebra *simple* if it has no non-trivial ideals and if its dimension is greater than 1.

Let $\mathfrak{g}$ be a Lie algebra. Then we can define the following concepts.

- The *lower central series*: $D_1\mathfrak{g} \supset D_2\mathfrak{g} \supset \ldots \supset D_k\mathfrak{g} \supset \ldots$ where D_k is defined by $D_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $D_k\mathfrak{g} = [\mathfrak{g}, D_{k-1}\mathfrak{g}]$. Thus

  $$[\mathfrak{g}, \mathfrak{g}] \supset [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supset [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \supset \ldots$$

  We call the Lie algebra $\mathfrak{g}$ *nilpotent* if $D_k\mathfrak{g} = 0$ for some $k \in \mathbb{N}$.

- The *upper central series*: $D^1\mathfrak{g} \supset D^2\mathfrak{g} \supset \ldots \supset D^k\mathfrak{g} \supset \ldots$ where D^k is defined by $D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$. Thus

  $$[\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supset \ldots$$

  We call the Lie algebra $\mathfrak{g}$ *solvable* if $D^k\mathfrak{g} = 0$ for some $k \in \mathbb{N}$.

Note that $D^k\mathfrak{g} \subset D_k\mathfrak{g}$ for all k. (proof: $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \to [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \subset [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$, etc.)
Thus nilpotency implies solvability.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* of $\mathfrak{g}$ if it is nilpotent and moreover, there does not exist a proper subalgebra of $\mathfrak{g}$, other than $\mathfrak{h}$ itself, that contains $\mathfrak{h}$ as an ideal.

Let the field $\mathbb{K}$ be algebraically closed and be of characteristic zero. Then each simple Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is a one dimensional subspace for all $\alpha \in \Phi$ and where $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$. Here Φ is the set of roots corresponding to the simple Lie algebra; the roots can be interpreted as the weights of the spaces $\mathfrak{g}_\alpha$ under the action on $\mathfrak{h}$. There now exists a nice basis for $\mathfrak{g}$, associated to this Cartan decomposition.

Definition 2.9 *The simple Lie algebra with root system* Φ *and fundamental set of roots* Π *has a* Chevalley basis

$$\{ x_\alpha \mid \alpha \in \Phi \} \cup \{ h_\alpha \mid \alpha \in \Pi \}$$

where $x_\alpha \in \mathfrak{g}_\alpha$ *for each* α *and* $h_\alpha := [x_\alpha, x_{-\alpha}]$ *such that*

1. $[x_\alpha, x_\beta] = 0$ *if* $\alpha + \beta \notin \Phi$*;*
2. $[x_\alpha, x_\beta] = \pm(p+1)x_{\alpha+\beta}$ *where* p *is the greatest integer such that* $\alpha - p\beta \in \Phi$.
3. $[h_\alpha, h_\beta] = 0$*;*
4. $[h_\alpha, x_\beta] = \frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle} x_\beta$*;*

Note that all of the *structure constants* are integral, hence the Lie algebra can be defined over $\mathbb{Z}$ and by tensoring (with an arbitrary field $\mathbb{K}$) it can be defined over any field $\mathbb{K}$. We call the elements x_α for $\alpha \in \Phi$ *root elements*.[2]

The above mentioned construction can also take place in a more general context, where the root datum is used.

Definition 2.10 *Let* $\mathcal{R} = (U, \Phi, Y, \Phi^*)$ *be a root datum. Then there is a Lie algebra with root system* Φ *and fundamental set of roots* Π *with a basis*

$$\{ x_\alpha \mid \alpha \in \Phi \} \cup \{ h_\alpha \mid \alpha \in \Pi \}$$

such that the basis elements satisfy conditions (1),(2) and (3) from Definition 2.9 and moreover also satisfy the following conditions:

1. $[x_\alpha, h_\beta] = \langle\alpha, \beta^*\rangle_\sharp x_\alpha$*,*
2. $[x_\alpha, x_{-\alpha}] = \sum_{i=1}^{7} \langle\omega_i, \alpha^*\rangle_\sharp h_{\alpha_i}$*, where* $\omega_1, \ldots, \omega_7$ *are such that they satisfy* $\langle\omega_i, \alpha_j^*\rangle_\sharp = \delta_{ij}$*. They are called the* fundamental weights*.*

According to Chevalley [9], again all structure constants are integral.

2.2.3 Lie groups and Chevalley groups

The map $\mathrm{ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ is defined for arbitrary $g \in \mathfrak{g}$ by $\mathrm{ad}_g h := [g, h]$ for all $h \in \mathfrak{g}$. For the root elements $x_\alpha(\alpha \in \Phi)$ the maps ad_{x_α} (which are nilpotent), give rise to automorphisms of the Lie algebra,

$$\exp t\mathrm{ad}_{x_\alpha} \colon g \mapsto g + t[x_\alpha, g] + t^2/(2!)[x_\alpha, [x_\alpha, g]] + \ldots$$

for all $g \in \mathfrak{g}$.

Definition 2.11 *The* Chevalley group G *corresponding to the Lie algebra* $\mathfrak{g}$*, is the automorphism group of* $\mathfrak{g}$ *and is generated by* $\{ X_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{K} \}$*, where* $X_\alpha(t) := \exp t ad_{x_\alpha}$*. If* $\mathbb{K}$ *has* characteristic 0*, we call* G *a Lie group.*

[2] More generally, the term root elements is also used to denote the elements from the $\mathrm{Aut}(\mathfrak{g})$-orbit of x_α for $\alpha \in \Phi$

The subgroups $\langle \{ X_\alpha(t) \mid t \in \mathbb{K} \}, \{ X_{-\alpha}(t) \mid t \in \mathbb{K} \} \rangle$ are called the root subgroups.

The Lie group G has a subgroup that can be related to the Weyl group W of the root system Φ. Define $n_\alpha := n_\alpha(1)$ and $h_\alpha(t) := n_\alpha(t)n_\alpha(-1)$, where

$$n_\alpha(t) := X_\alpha(t)X_{-\alpha}(-t^{-1})X_\alpha(t) \qquad \text{for all } \alpha \in \Phi.$$

Now let N be the subgroup of G, generated by all n_α ($\alpha \in \Phi$) and all $h_\alpha(t)$ ($\alpha \in \Phi, t \in \mathbb{K}$). Let H be the subgroup of N, generated by all $h_\alpha(t)$ ($\alpha \in \Phi, t \in \mathbb{K}$). The elements n_α act in the same way on $\mathfrak{h}$ as $w_\alpha \in W$ do. Moreover, they permute the root spaces $\{ \mathfrak{g}_\beta \mid \beta \in \Phi \}$ in the same way that w_α permute the corresponding roots $\{ \beta \mid \beta \in \Phi \}$. We have the following lemma.

Lemma 2.5 *There is an isomorphism of groups between W and N/H, given by $w_\alpha \mapsto n_\alpha H$, for all $\alpha \in \Phi$. Let $\dot{n}_\alpha$ denote an arbitrary representative of the coset $n_\alpha H$.*

We call the elements n_α ($\alpha \in \Pi$), fundamental reflections, just like the elements w_α, $\alpha \in \Pi$.

2.3 An introduction to character theory

We introduce some character theory that is used in the remainder of this thesis. For a more detailed treatment of the theory of characters we refer to the very nice and readable books [20] and [22].

Given a group G acting on a set X and a representation ρ of G, the character χ corresponding to this representation ρ is defined to be the function $\chi\colon G \to \mathbb{C}$ that maps every $g \in G$ to $\mathrm{Tr}(g\rho)$. The function $\mathrm{Tr}(\cdot)$ is the well known trace function of a matrix, returning the sum of the diagonal entries in an arbitrary representation. The easiest example of a character is the trivial character 1_G which corresponds to the trivial linear representation mapping all elements $g \in G$ to the element $(1) \in \mathrm{GL}(1, \mathbb{C})$. Characters belong to the more generally defined space of *class functions*. A class function is a function from a group G to $\mathbb{C}$ that is constant on conjugacy classes. Our interest lies with characters and more in particular, in permutation characters. A permutation character π is a character corresponding to the permutation action of a certain group G on a set X. Let ρ_π be the permutation representation of G. Then the matrices $g\rho_\pi$ ($g \in G$) will be permutation matrices and we obtain a useful property of permutation characters. For each $g \in G$, the corresponding character π is determined by

$$\pi(g) = \text{the number of fixed points of } X \text{ under the action of } g.$$

Let $H \subseteq G$ be a subgroup. For a character χ of G, we define $\chi|_H$ to be the restriction of χ to H, which is defined in the obvious way: $\chi|_H(g) = \chi(g)$ if $g \in H$. One nice property of characters is that this process can sort of less be inverted. Thus starting out with a character ϑ of H we introduce the induced character ϑ^G to be the corresponding character of G defined by

$$\vartheta^G(g) = \frac{1}{|H|} \sum_{x \in G} \vartheta^*(xgx^{-1}),$$

where ϑ^* is defined by $\vartheta^*(h) = \vartheta(h)$ if $h \in H$ and $\vartheta^*(g) = 0$ if $g \notin H$.

On the space of class functions in general and the subset of characters in particular there is the concept of an inner product.

Definition 2.12 *Let χ and ϑ be class functions on a group G. Then the inner product of χ and ϑ is given by*

$$\langle \chi, \vartheta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\vartheta(g)}$$

where $\bar{\cdot}$ represents complex conjugation.

The notation for this inner product is the same as the notation of the product on the root space. This should not lead to any confusion.

Definition 2.13 *In the vector space of class functions there exists a basis of characters* $\chi_1, \dots, \chi_k$ *which is orthonormal with respect to the inner product defined in Definition 2.12. This means that* $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ *for all* $i, j \in \{1, \dots, k\}$*. A character* χ *is called irreducible if it is the trace function of an irreducible representation.*

Now χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

We state a few lemmas about characters without proof. The lemma's including their proofs can be found in [20]. We start by introducing the Frobenius reciprocity which will prove to be very useful later on.

Lemma 2.6 *Let* $H \subseteq G$ *and suppose that* χ *and* ϑ *are characters (or more generally class functions) corresponding to* G *respectively* H*. Then*

$$\langle \chi|_H, \vartheta \rangle = \langle \chi, \vartheta^G \rangle$$

where $\langle \cdot, \cdot \rangle$ *is the usual inner product defined on the space of class functions.*

The next lemma shows a useful connection between computations with characters, and orbits.

Lemma 2.7 *Let* G *act on a* G*-module* X *with permutation character* χ*. Then* $\langle \chi, 1_G \rangle$ *equals the number of* G*-orbits on* X*. This number is called the* permutation rank *of* χ*.*

3 An introduction to distance-transitive graphs

The main purpose of this thesis is to deliver a contribution to the classification of distance-transitive graphs. Remember that a graph $\Gamma = \Gamma(V, E)$ is distance-transitive if its automorphism group G acts transitively on all of its distance sets $D_i(\Gamma)$. It is possible to describe Γ in terms of groups.

3.1 From graphs to groups

Let Γ be a distance-transitive graph. First fix a vertex $v \in V$. Then obviously G acts transitively on all sets $\{ (v, w) \mid w \in V, \mathrm{d}(v, w) = i \}$, hence G_v acts transitively on all distance sets $D_i(v)$. Define $H := G_v$. Finally let $r \in G$ such that $v^r \in D_1(v)$. Then G, H and r fully determine the graph Γ. We make this concrete.

Definition 3.1 *Let* $\Gamma' := \Gamma'(G, H, r)$ *be the graph with vertex set* $H\backslash G$ *in which two vertices* Hx *and* Hy *are adjacent if and only if* $y \in HrHx$*.*

The theorems in this section are elaborate versions of theorems in [10].

Theorem 3.1 *The graphs* Γ *and* Γ' *are isomorphic.*

Proof: Clearly $V = \{ v^g \mid g \in G \} = \{ v^{Hg} \mid Hg \in H\backslash G \}$ thus there is a one to one correspondence between elements $v^{Hg} \in V$ and $Hg \in H\backslash G$. In particular, the element v corresponds to H. By definition, for two vertices $Hx, Hy \in H\backslash G$, we have $Hx \sim Hy$ exactly when $v^{Hx} \sim v^{Hy}$. Since $v \sim v^{rh}$ for all $h \in H$, we see that $v^{h_1 x} \sim v^{rhh_1 x}$ for all $h \in H$ and hence $v^{h_1 x} \sim v^{Hrh'x}$ for all $h' \in H$, hence $Hx \sim Hy$ if and only if $Hy \in HrHx$, hence if and only if $y \in HrHx$. □

This way of describing a distance-transitive graph, suggests that we check whether the opposite route can be followed as well; given a group G with a subgroup H, what are the conditions on G and H such that it is possible to construct a distance-transitive graph that is isomorphic to the graph $\Gamma'(G, H, r)$ for some $r \in G$? Many conditions exist (see [10]) and two of them are stated in the following theorem.

Theorem 3.2 *Let G be a group, with a subgroup H. Consider the graph Γ' as defined in 3.1 for some $r \in G$. Then*

- Γ' *is connected if and only if $\langle H, r\rangle = G$;*
- Γ' *is undirected if and only if the G-orbit on $H\backslash G \times H\backslash G$ containing (H, Hr) is self paired.*

Proof: The subgroup $\langle H, r\rangle$ is strictly smaller than G if and only if it does not work transitively on the set of right cosets of H in G. Thus if and only if it stabilizes some subset of $H\backslash G$. But this means that no vertex in this subset is connected to a vertex outside of the subset, hence that Γ' is not connected.

Next, a graph $\Gamma(V, E)$ is undirected if and only if the edge set is self paired. The edge set of Γ' is exactly the G orbit on $H\backslash G \times H\backslash G$ containing (H, Hr). □

Corollary 3.1 *For H a maximal subgroup of G, the graph $\Gamma'(G, H, r)$ is connected for all $r \in G, r \notin H$ such that $r^{-1} \in HrH$.*

3.2 Towards a classification

We introduce the concept of *primitivity* of a graph. For this we need to introduce *blocks* first. A block B of a vertex set V under the action of G is a subset of V such that B and B^g are either disjoint or identical for all $g \in G$. It is called trivial if $B = \emptyset$, $|B| = 1$ or $B = V$. The group G is called *primitive* if the only blocks are trivial ones. It is called *imprimitive* if there exists a non trivial block.

Example:
Consider the cube-graph in Figure 2. Let G be its automorphism group. It is easy to verify that the vertex set $\{1, \dots, 8\}$ can be partitioned in the 4 blocks $\{1, 7\}$, $\{2, 8\}$, $\{3, 5\}$ and $\{4, 6\}$ under the action of G. Moreover, there is yet another partitioning into the two blocks $\{1, 3, 6, 8\}$ and $\{2, 4, 5, 7\}$. Hence G is imprimitive and according to the following definition, the cube graph is imprimitive.

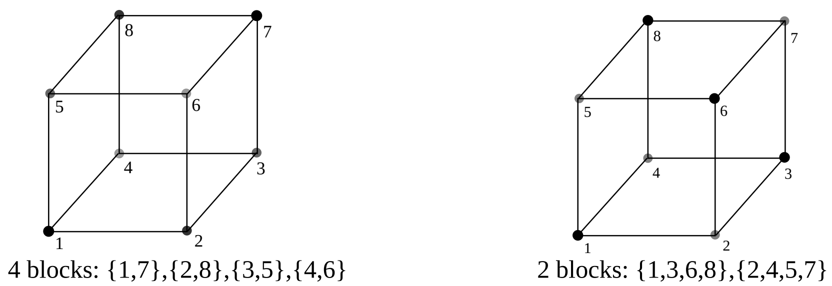


Figure 2: Partitioning of the vertex set into two respectively 4 blocks.

Definition 3.2 *A graph is called* (im)primitive *when its automorphism group is (im)primitive.*

The next theorem, which is due to Smith [28], is very useful in the classification of distance-transitive graphs.

Theorem 3.3 *An imprimitive distance-transitive graph Γ with degree $k \geq 3$ is either bipartite or antipodal (or both).*

Theorem 3.4 *Suppose that G acts distance-transitively on a graph Γ with diameter d.*

1. *If Γ is antipodal, then G acts distance-transitively on the* antipodal quotient *of Γ, whose vertices are the equivalence classes of $D_0(\Gamma) \cup D_d(\Gamma)$ and in which two vertices are adjacent whenever they contain adjacent vertices in Γ.*

2. *If Γ is bipartite, then G acts distance-transitively on the* halved graphs *of Γ, which are the two graphs obtained from Γ by taking one of the two bipartite classes and letting two vertices be adjacent if they are at distance 2 in Γ.*

Proof: See Theorem 4.1.10 in [5]. □

Example:
In the previous example, we saw that the vertex set of the cube graph consists of two blocks, hence it is imprimitive. The cube graph obviously is distance-transitive. Thus Theorem 3.3 tells us that it is bipartite or antipodal. In fact it is both. Thus we can apply Theorem 3.4. By taking the antipodal quotient or by halving the graph, we obtain smaller distance-transitive graphs (see Figure 3). Note that in this example the antipodal quotient and the halved graph(s) are isomorphic. They

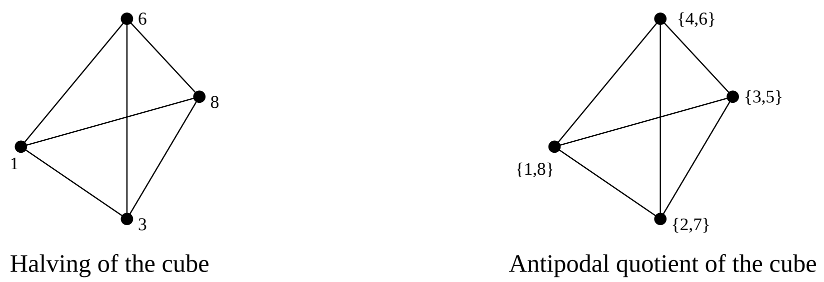


Figure 3: Primitive distance-transitive graphs, obtained from the cube graph

are both tetrahedra. The tetrahedron is easily seen to be a primitive distance-transitive graph. Thus the primitive distance-transitive graphs can be considered to be the building blocks of all distance-transitive graphs. The key to the classification of all distance-transitive graphs lies in classifying these building blocks. The next theorem gives us more information on the groups related to these primitive distance-transitive graphs.

Theorem 3.5 *Let Γ be a graph with automorphism group G, which acts transitively on the vertex set of Γ. The graph Γ is primitive if and only if the stabilizer G_x is a maximal subgroup of G for each x in the vertex set of Γ.*

Proof: This theorem follows directly from Theorem 9.15 in [27]. □

Thanks to the classification of all simple groups, there exists a finite[3] list of pairs of groups and corresponding maximal subgroups that might give rise to a primitive distance-transitive graph. There is hope that all *primitive* distance-transitive graphs will be classified. A lot of work has already been done. See [10] for a good overview. Also [5] is a good reference for known distance-transitive graphs, or more generally, distance-regular graphs. Lists of both the open cases, and the cases that are already dealt with can be found on the website [11]. For the open cases, it remains to be shown that a related distance-transitive graph does (not) exist. In this thesis we investigate one of these cases, namely that of an automorphism group $E_7(q)$.

[3]This list is only finite if we allow parameters in the entries.

4 Graphs related to $E_7(q)$

As mentioned in section 3.2, a lot of work has already been done to classify all primitive distance-transitive graphs. Especially amongst the exceptional Lie groups there are still some hard open cases. One of them is the question whether there exists a distance-transitive graph with automorphism group $E_7(q)$. In section 3.1 we have seen that the maximal subgroups of $E_7(q)$ play the role of a vertex stabilizer in such a graph. Hence we first need to know more about the maximal subgroups of $E_7(q)$.

4.1 'Large' maximal subgroups

In case the maximal subgroup is parabolic, everything is known and an overview of the results can be found in [5]. Hence we limit ourselves to the non-parabolic case. Furthermore, there is a lower bound on the size of the maximal subgroup. The following theorem comes from [24].

Theorem 4.1 *Let G be a simple Chevallley group with a Borel subgroup B and let W be the corresponding Weyl group. Furthermore, let H be a maximal subgroup of G. If there exists a distance-transitive graph with automorphism group G and vertex stabilizer H, then $|H| > |G : B|/|W|$.*

Hence the only maximal subgroups that are eligible to function as a vertex stabilizer in a distance-transitive graph, are the ones of small index. In our case this means that we are only interested in maximal subgroups H of size $|H| > q^{63}$ for $q > 2$ or $|H| > q^{49}$ for $q = 2$. A list of large maximal subgroups for all exceptional groups is derived in [23]. Note that this reference does not contain the most up-to-date information; in the case of $E_7(q)$, [23] only mentions the maximal subgroups with an order exceeding q^{64}. The table in [11], contains all non-parabolic maximal subgroups of $E_7(q)$ with an order exceeding q^{63}. From this table, we obtain the information in Table 1, which contains all non-parabolic maximal subgroups of $E_7(q)$ that might function as the vertex stabilizer in a distance-transitive graph, with automorphism group $E_7(q)$.[4]

Maximal subgroup of $E_7(q)$	DTG
$N(E_6(q)) = (E_6(q)T_{q-1}) \cdot (q-1,3) \cdot 2$	?
$N(^2E_6(q)) = (^2E_6(q)T_{q+1}) \cdot (q+1,3) \cdot 2$	?
$N(A_1(q)D_6(q))$	no
$E_7(q^{1/2})$	no
$N(A_7(q)) = A_7(q) \cdot 2,\ q = 2, 4$	no
$N(^2A_7(q)) = {}^2A_7(q)$	?

Table 1: The 'large' maximal subgroups of $E_7(q)$

A case-by-case study needs to be performed for the open cases in Table 1. In the remainder of this thesis we contribute to this work. In accordance with Theorem 3.1, we prove that the graph $\Gamma'(E_7(q), A_7(q) \cdot 2, r)$ is not distance-transitive for any $r \in E_7(q)$, $r \notin A_7(q) \cdot 2$, where all groups are defined over the field GF(q) of characteristic 2.

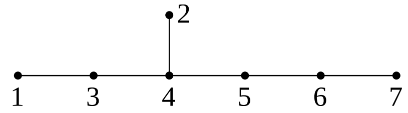
4.2 Construction of the maximal subgroup $A_7(q) \cdot 2$ of $E_7(q)$

First the Chevalley groups $A_7(q)$ and $E_7(q)$ are introduced and their subgroup relation is shown. Next the symmetry of the Dynkin diagram of type A_7 is used to construct the maximal subgroup $A_7(q) \cdot 2$ of $E_7(q)$. In this section knowledge of some basic concepts from Lie theory, like root systems, Dynkin diagrams, Chevalley groups and Lie algebras, is required. We refer to section 2.2 for a short introduction into Lie theory.

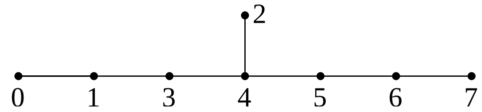
[4] It is currently still unclear to the author, why one is not interested in groups H over GF(2) such that $2^{49} < |H| < 2^{63}$.

4.2.1 Construct $A_7(\mathbb{K})$ as a subgroup of $E_7(\mathbb{K})$

Consider the group of Lie type E_7 over some field $\mathbb{K}$, for which the root datum has adjoint isogeny type (see e.g., [12],[30]); we call this group $E_7(\mathbb{K})$. This group is characterized by the following Dynkin diagram[5]. The nodes $i = 1, \ldots, 7$ in this diagram correspond to the fundamental roots $\alpha_1, \ldots, \alpha_7 \in \Phi$ where Φ is the root system of type E_7.



From the Dynkin diagram we construct the so-called *extended diagram*; there exists a root $\alpha_0 \in \Phi$ that is (of course) linearly dependent on the fundamental roots $\alpha_1, \ldots, \alpha_7$ such that the Dynkin diagram can be extended to the following one:



We compute coefficients $a_1, \ldots, a_7$ such that $\alpha_0 = \sum_{i=1}^{7} a_i\alpha_i$ satisfies the relations on the roots imposed by the extended Dynkin diagram above. In this way we prove the existence of this root α_0.

From the extended Dynkin diagram, it follows that the root α_0 should have zero inner product with the roots $\alpha_2, \ldots, \alpha_7$. All inner products between fundamental roots are imposed by the Dynkin diagram through the following set of rules

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j; \\ 0 & \text{if node } i \text{ and node } j \text{ are not connected;} \\ -1 & \text{if node } i \text{ and node } j \text{ are connected.} \end{cases}$$

In order to find the coefficients, we first set $a_7 = \xi$. Then we easily obtain values for $a_1, \ldots, a_6$ expressed in ξ.

$$0 = (\alpha_0, \alpha_7) = a_6(\alpha_6, \alpha_7) + a_7(\alpha_7, \alpha_7) = -a_6 + 2a_7 \quad \rightarrow \quad a_6 = 2\xi$$

and similarly

$$\begin{array}{lcl} 0 = (\alpha_0, \alpha_6) = a_5(\alpha_5, \alpha_6) + a_6(\alpha_6, \alpha_6) + a_7(\alpha_7, \alpha_6) & \rightarrow & a_5 = 3\xi \\ 0 = (\alpha_0, \alpha_5) = a_4(\alpha_4, \alpha_5) + a_5(\alpha_5, \alpha_5) + a_6(\alpha_6, \alpha_5) & \rightarrow & a_4 = 4\xi \\ 0 = (\alpha_0, \alpha_2) = a_2(\alpha_2, \alpha_2) + a_4(\alpha_4, \alpha_2) & \rightarrow & a_2 = 2\xi \\ 0 = (\alpha_0, \alpha_4) = a_2(\alpha_2, \alpha_4) + a_3(\alpha_3, \alpha_4) + a_4(\alpha_4, \alpha_4) + a_5(\alpha_5, \alpha_4) & \rightarrow & a_3 = 3\xi \\ 0 = (\alpha_0, \alpha_3) = a_1(\alpha_1, \alpha_3) + a_3(\alpha_3, \alpha_3) + a_4(\alpha_4, \alpha_3) & \rightarrow & a_1 = 2\xi \\ -1 = (\alpha_0, \alpha_1) = a_1(\alpha_1, \alpha_1) + a_3(\alpha_3, \alpha_1) & \rightarrow & \xi = -1 \end{array}$$

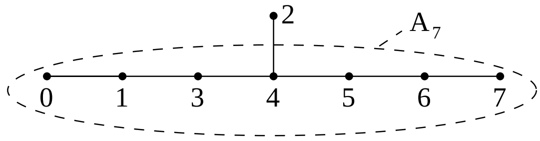
Thus we obtain $\alpha_0 = -(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$. There is one more relation that is imposed by the extended Dynkin diagram, namely that $(\alpha_0, \alpha_0) = 2$. But we easily show that

$$(\alpha_0, \alpha_0) = a_1(\alpha_1, \alpha_0) = -2a_1 = 2$$

thus also this relation is satisfied. Note that $a_0 \in \Phi^-$. The root $-a_0 \in \Phi^+$ is known as the *highest (long) root* of E_7.

It is not hard to see how the Dynkin diagram A_7 appears in the extended diagram E_7 as the subdiagram on the nodes $0, 1, 3, 4, 5, 6, 7$. This implies that in order to obtain $A_7(\mathbb{K})$ within $E_7(\mathbb{K})$, we restrict ourselves to the roots $\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$.

[5] The numbering of the nodes is in accordance with Bourbaki's convention. See [4].



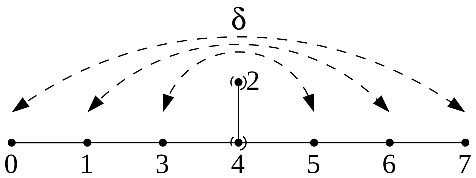
More formally, we consider the so-called smallest *closed set of roots* $\Psi \subset \Phi$ which contains these roots $\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ (see A). Ψ is a root system of type A_7. The group $A_7(\mathbb{K})$ is generated by the root subgroups X_α for $\alpha \in \Psi$. See also appendix A.

In the remainder of this thesis we limit ourselves to the case where $\mathbb{K} = \mathrm{GF}(q)$ for $q = 2^p$ for some p. We also use the notation $\mathbb{F}_q$ for $\mathrm{GF}(q)$. We define $G := E_7(q)$. Moreover, we define L_{E_7} to be the Lie algebra with root system Φ over $\mathrm{GF}(q)$ of adjoint isogeny type (see Section 4.3). Let L_{A_7} be the subalgebra of L_{E_7} of type A_7 that is obtained from the root system Ψ.

4.2.2 Using the diagram automorphism to obtain $A_7(q) \cdot 2$

We just constructed $A_7(q)$ within $E_7(q)$. However, we are not directly interested in the subgroup $A_7(q)$ of $E_7(q)$, but in the maximal subgroup $A_7(q) \cdot 2$ of $E_7(q)$. This group is twice as big and is obtained from $A_7(q)$ by adding a graph automorphism on top of $A_7(q)$.

The extended Dynkin diagram of E_7 is clearly symmetrical; accordingly there exists a diagram automorphism δ of the extended Dynkin diagram.



One easily sees that δ is an involution. It induces a permutation of Φ determined by its right action on the fundamental roots and on α_0; $\alpha_0\delta = \alpha_7, \alpha_1\delta = \alpha_6, \alpha_2\delta = \alpha_2, \alpha_3\delta = \alpha_5, \alpha_4\delta = \alpha_4, \alpha_5\delta = \alpha_3, \alpha_6\delta = \alpha_1$, and $\alpha_7\delta = \alpha_0$.

The diagram automorphism δ induces a graph automorphism σ_δ of group elements. The action of σ_δ on the group elements is defined by the action on the generators;

$$\sigma_\delta^{-1} X_\alpha(t) \sigma_\delta := X_{\alpha\delta}(t) \text{ for every } \alpha \in \Phi.$$

Our definition of a graph automorphism differs only from the one in Carter ([7]) in that we use right actions and that we work over a field of characteristic 2, hence the sign of t in the above equation is irrelevant. Now let $H := A_7(q) \cdot 2$ be the group generated by $A_7(q)$ and σ_δ.

There is also another way to interpret the extension of $A_7(q)$. Let w_0^Ψ and w_0^Φ be the longest elements of the Weyl groups $W(\Psi)$ and $W(\Phi)$ respectively. It can easily be seen that $w_0^\Phi \colon \alpha \mapsto -\alpha$ for all $\alpha \in \Phi$, since all reflections leave the relative position of the roots invariant and the root system Φ has no symmetries.

Note that due to the symmetry of Ψ there are exactly two candidates for the longest Weyl group element w_0^Ψ; either $w_0^\Psi \colon \alpha \mapsto -\alpha$ or $w_0^\Psi \colon \alpha \mapsto -\alpha\delta$ for each $\alpha \in \Psi$. The former option implies that w_0^Ψ is the product of an odd number of reflections, but since the length of a reflection equals the number of positive roots it maps to negative roots, it should also be the product of $|\Psi^+| = 28$ reflections. This is a contradiction hence $w_0^\Psi \colon \alpha \mapsto -\alpha\delta$ for all $\alpha \in \Psi$.

Thus $\delta = w_0^\Psi w_0^\Phi$. Now note that $w_0^\Psi \in W(\Psi) \subset A_7(q)$ thus H is generated by $A_7(q)$ and σ_{w_0}, where σ_{w_0} is the graph automorphism induced by $w_0^\Phi|_\Psi$ in a similar way as above, thus $\sigma_{w_0}^{-1} X_\alpha(t) \sigma_{w_0} := X_{\alpha w_0^\Phi}(t)$ for every $\alpha \in \Psi$.

4.3 The structure of L_{E_7} of adjoint isogeny type

Over a field of small characteristic, there might exist more Lie algebras of the same Cartan type, that are mutually non isomorphic. This is related to the fact that the corresponding root data can have different isogeny type. First we give a small example, showing three different Lie algebras of type A_1 in characteristic 2. Next we look at the Lie algebra of type E_7, corresponding to a root datum of adjoint isogeny type and we study its structure. More information on the isogeny type of root data can be found in [30] and [8].

4.3.1 An example: the different appearances of L_{A_1} over $GF(q)$

When we study the Lie algebra L_{A_1} over $\mathbb{C}$, we know from theory ([18],[32],[31]) that its Chevalley basis $\mathcal{B}_1$ is given by x, y, h that satisfy the following commutator relations.

$$\begin{aligned}[x,y] &= h\\ [h,x] &= 2x\\ [h,y] &= -2y\end{aligned}$$

Still considering L_{A_1} over $\mathbb{C}$ we can perform basis transformations yielding similar bases. For example, we can perform the substitutions $x' = x$, $y' = y$, $h' = h/2$, giving us the basis $\mathcal{B}_2$ such that the basis elements satisfy the commutator relations:

$$\begin{aligned}[x',y'] &= 2h'\\ [h',x'] &= x'\\ [h',y'] &= -y'\end{aligned}$$

Yet another substitution we can perform is $x'' = x$, $y'' = y/2$, $h'' = h/2$, giving us a basis $\mathcal{B}_3$ such that the basis elements satisfy the commutator relations:

$$\begin{aligned}[x'',y''] &= h''\\ [h'',x''] &= x''\\ [h'',y''] &= -y''\end{aligned}$$

Note that all the multiplication constants are integral. Hence for each of these bases, we can construct a Lie algebra of type A_1 over the ring $\mathbb{Z}$, by taking all linear combinations of the basis elements of $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ respectively, with coefficients in $\mathbb{Z}$. Let $L_i(\mathbb{Z})$ be the Lie algebra over $\mathbb{Z}$ with the basis $\mathcal{B}_i$ for $i = 1, 2, 3$, constructed in such a way.

We next construct the Lie algebras L_1,L_2, and L_3 over the field GF(q) from $L_1(\mathbb{Z})$,$L_2(\mathbb{Z})$, and $L_3(\mathbb{Z})$, respectively. Remember that here $q = 2^p$ for some p, hence GF(q) is a field of characteristic 2. The construction makes use of tensoring and is explained well in [7]. We define

$$L_i := L_i(\mathbb{Z}) \otimes \mathrm{GF}(q) \text{ for } i = 1, 2, 3.$$

The basis $\mathcal{B}'_i$ of L_i is now defined by $\mathcal{B}'_i = \{ b \otimes 1_{\mathrm{GF}(q)} \,|\, b \in \mathcal{B}_i \}$ for $i = 1, 2, 3$. The Lie multiplication between two elements $x \otimes 1_{\mathrm{GF}(q)}$ and $y \otimes 1_{\mathrm{GF}(q)}$ in L_i is given by

$$[x \otimes 1_{\mathrm{GF}(q)}, y \otimes 1_{\mathrm{GF}(q)}] = [x, y] \otimes 1_{\mathrm{GF}(q)}.$$

It follows that the multiplication constants of L_i can be interpreted as elements from the prime subfield GF(2) of GF(q). Hence the commutator relations of L_1, L_2, and L_3 simplify to:

L_1	L_2	L_3
$[x,y] = h$	$[x',y'] = 0'$	$[x'',y''] = h''$
$[h,x] = 0$	$[h',x'] = x'$	$[h'',x''] = x''$
$[h,y] = 0$	$[h',y'] = y'$	$[h'',y''] = y''$

Note that the Lie algebras obtained here are all three essentially different. L_1 contains a one dimensional center consisting of the basis element h and has decomposition structure $\frac{2}{1}$. In L_2 the elements x' and y' generate an ideal and L_2 has decomposition $\frac{1}{2}$. Finally, L_3 is simple. Thus these three Lie algebras are not isomorphic over $GF(q)$.

4.3.2 Analogy between Lie algebras of type E_7 and A_1

As seen in the previous section for the L_{A_1}-case, working with Lie algebras over finite fields with small characteristic can cause some anomaly; different representations of the same Lie algebra might give rise to different structures when considered over a field of small characteristic. Let us consider Lie algebras L_{E_7} over GF(q).

There are more ways to construct a Lie algebra, related to the isogeny type of the corresponding root datum (see [30],[8]). One can create a so-called *Simply Connected* version or an *Adjoint* version. The Simply Connected version of creating a Lie algebra results in a Lie algebra with the standard Chevalley basis (which we call L_{SC}). Since all multiplication constants are integral, we can consider L_{SC} over GF(q).

We now explicitly construct the Adjoint version L_{Ad} from the root datum. We study the commutator relations of the basis elements of L_{Ad} and we obtain a 1/132-structure of L_{Ad} over GF(q).

In Magma , we constructed the Lie algebra L_{Ad} explicitly over the field GF(2). In appendix C.2 we use Magma to check that this construction indeed returns a Lie algebra.

Consider the root datum $\mathcal{R} = (U, \Phi, Y, \Phi^*)$, for U and Y free $\mathbb{Z}$-modules and Φ a root system of type E_7. Remember that there is a bilinear pairing $\langle\, \cdot, \cdot\, \rangle_\sharp$ between U and Y, putting them in duality (see section 2.2.1). The fundamental roots are $\alpha_1, \dots, \alpha_7 \in \Phi$, where α_i is the i'th unit vector of U. The corresponding fundamental coroots, with respect to the standard basis $y_1, \dots, y_7$ of Y, are:

$$\begin{aligned}
\alpha_1^* &= (\,2\,0\,-1\,0\,0\,0\,0\,);\\
\alpha_2^* &= (\,0\,2\,0\,-1\,0\,0\,0\,);\\
\alpha_3^* &= (\,-1\,0\,2\,-1\,0\,0\,0\,);\\
\alpha_4^* &= (\,0\,-1\,-1\,2\,-1\,0\,0\,);\\
\alpha_5^* &= (\,0\,0\,0\,-1\,2\,-1\,0\,);\\
\alpha_6^* &= (\,0\,0\,0\,0\,-1\,2\,-1\,);\\
\alpha_7^* &= (\,0\,0\,0\,0\,0\,-1\,2\,).
\end{aligned}$$

Now let L_{Ad} be the Lie algebra with root system Φ and basis elements

$$\{\, x_\alpha \mid \alpha \in \Phi^+ \,\} \cup \{\, h_{y_i} \mid i = 1, \dots, 7 \,\} \cup \{\, x_{-\alpha} \mid \alpha \in \Phi^+ \,\}$$

such that the relations in Definition 2.10 are satisfied. For every α, the element $h_\alpha = \sum_{i=1}^{7} \lambda_i h_{y_i}$ whenever the corresponding coroot $\alpha^* = \sum_{i=1}^{7} \lambda_i y_i$. According to Chevalley [9], all the structure constants are integral, hence we can consider the Lie algebra over any field. In the remainder of this section we assume that L_{Ad} is a Lie algebra over GF(q).

One easily checks that the *Cartan matrix* $(\langle \alpha_i, \alpha_j^* \rangle_\sharp)_{1 \le i,j \le n}$, the rows of which are exactly the fundamental coroots, has determinant 2, hence the coroots form a dependent set when considered over a field of characteristic 2. In particular we see that $\alpha_2^* + \alpha_5^* + \alpha_7^* = (0\,2\,0\,-2\,2\,-2\,2)$, hence $h_{\alpha_2} + h_{\alpha_5} + h_{\alpha_7} = 2h_{y_2} - 2h_{y_4} + 2h_{y_5} - 2h_{y_6} + 2h_{y_7} = 0$ since GF(q) is a field of characteristic 2.

So the root elements generate a Lie algebra L_0 of dimension 132. Clearly L_0 is a subalgebra of L_{Ad}. The element $v := h_{y_2}$ is easily seen to be in the complement of L_0 in L_{Ad}. Since v is a Cartan element, it commutes with all other Cartan elements. Moreover, it commutes with the elements from $\langle \{\, x_\alpha \mid \alpha \in \Psi \,\} \rangle$ and it stabilizes the elements from $\{\, x_\alpha \mid \alpha \in \Phi \setminus \Psi \,\}$. Thus clearly

$[L_0, v] \subseteq L_0$. But then also $[L_0, L_{Ad}] = [L_0, \langle v, L_0\rangle] \subseteq L_0$ so L_0 is an ideal of L_{Ad}. Hence L_{Ad} has a 1/132-structure over GF(q). Whenever we write L_{E_7} we refer to the Lie algebra L_{Ad} of adjoint isogeny type.

4.4 Construction of the graph $\Gamma \cong \Gamma'(E_7(q), A_7(q) \cdot 2, r)$

In section 3.1 the graph structure originating from a group G and a maximal subgroup H has as vertex set X' the set of right cosets of H in G, $X' = \{ Hg \mid g \in G \}$. By letting an element $r \in G, r \notin H$ determine adjacency $H \sim Hr$, we obtain a graph Γ' from this general graph structure. In this manner we can construct a graph $\Gamma'(E_7(q), A_7(q) \cdot 2, r)$ for arbitrary $r \in E_7(q)$ and $r \notin A_7(q) \cdot 2$. However, this graph is not pleasant to work with. The definition of the vertex set X' is a very general one, that does not use any special structure that occurs in our case, where $G = E_7(q)$ and $H = A_7(q) \cdot 2$.

Consider the Lie algebra element $v \in L_{E_7}$, defined in section 4.3. Clearly $\langle v\rangle \subseteq C_{L_{E_7}}(L_{A_7})$. Using Magma , we have seen that $\langle v\rangle = C_{L_{E_7}}(L_{A_7})$. We define the vertex set X of a graph Γ by $X := \{ \mathbb{F}_q v^g \mid g \in H\backslash G \}$. There is a clear one to one correspondence between X' and X given by $g \in X' \leftrightarrow \mathbb{F}_q v^g$. The elements of X can be seen as one dimensional vector spaces with respect to the Chevalley basis of L_{E_7}. Then X is a 133 dimensional vector space over the field GF(q) and $|X| = |G|/|H| \approx q^{133}/q^{63} = q^{70}$.

Now adjacency in the graph Γ is fully defined by selecting the H-orbit of v^r for some $r \in G$, $r \notin H$ as the neighbour set of v. Let $w \in X$. Then $|Hw| = |H|/|\mathrm{Stab}_H(w)|$ thus its size is bounded from above by $|H| \approx q^{63}$. This means that the number of distinct H-orbits on X is bounded from below by $|X|/|H| \approx q^7$. These numbers indicate that even for small values of q, this problem is much too big for a computation of all orbits and for an exhaustive computer search of a possible distance-transitive graph structure that originates from $E_7(q)$ and $A_7(q) \cdot 2$.

5 The graph Γ cannot be distance-transitive

In this section we use the theory of characters to conclude that there cannot exist a distance-transitive graph Γ with automorphism group G and a vertex set X, as defined in section 4.4. Recall that X is the G-orbit of the unique one dimensional Lie algebra $\mathbb{F}_q v$ for which $v = C_{L_{E_7}}(L_{A_7})$. Also remember that H is the stabilizer of $\mathbb{F}_q v$. Thus we can consider $\mathbb{F}_q v$ as an H-module. Now let π be the permutation character of the action of G on X. The subgroup H also acts on X. The corresponding permutation character is 1_H which denotes the identity function on H. Then $\pi = (1_H)^G$ (see [20]).

Theorem 5.1 *If the graph Γ with automorphism group G and vertex set X is distance-transitive, then the permutation character π of the G-action on X is multiplicity free. Moreover, if Γ has diameter d, then $\langle \pi, \pi\rangle = d + 1$.*

Proof: If Γ has diameter d, then there exists a partitioning of X into $d+1$ distance sets with respect to the element $v \in X$. Since Γ is assumed to be distance-transitive, the point stabilizer H of v acts transitively on all the distance sets, hence they correspond to $d + 1$ distinct H-orbits on X. Thus using Frobenius reciprocity and Lemma 2.7 we see that $\langle \pi, \pi\rangle = \langle \pi, (1_H)^G\rangle = \langle \pi|_H, 1_H\rangle = d + 1$. Now let $\chi_0, \ldots, \chi_d$ be the characters corresponding to the $d+1$ different H-orbits on X. (Note that we do not assume these characters to be distinct at this point.) Thus we can write $\pi = \sum_{i=0}^{d} \chi_i$. Then

$$d + 1 = \langle \pi, \pi\rangle = \sum_{i=0}^{d}\sum_{j=0}^{d}\langle \chi_i, \chi_j\rangle \geq \sum_{i=0}^{d}\langle \chi_i, \chi_i\rangle \geq d + 1$$

from which it follows that all the characters $\chi_i, i = 0, \ldots, d$ are distinct and irreducible. □

Thus we only need to show that the character π is not multiplicity free. We define a second G-set Y, by $Y := \{ \mathbb{F}_q(x_{\alpha_0})^g \mid g \in G \}$. Note that Y is the set of all (conjugate) root elements of L_{E_7}.

Let K be the stabilizer in G of $\mathbb{F}_q x_{\alpha_0}$. Now 1_K is the permutation character corresponding to the trivial action of K on Y. Similarly as before ρ is the permutation character of the action of G on Y. Again $\rho = (1_K)^G$ holds.

Theorem 5.2 *ρ is multiplicity free of rank 5.*

Proof: By applying Frobenius reciprocity and Lemma 2.7 again, we obtain $\langle \rho, \rho \rangle = \langle \rho, (1_K)^G \rangle = \langle \rho|_K, 1_K \rangle = n$, where n is the number of K-orbits on Y. The set Y can easily be identified with the set of right cosets $\{ Kg \mid g \in G \}$. Hence the number of K orbits on Y equals the number of double cosets $K \backslash G / K$. Since G is a Chevalley group it has a (B, N) pair and, using Bruhat decomposition, we can write $G = BNB$. Since K is a parabolic subgroup of G of type D_6, we can write $K = BN_{D_6}B$ for some subgroup N_{D_6} of N. Thanks to knowledge on parabolic groups we have $K \backslash G / K = BN_{D_6}B \backslash BNB / BN_{D_6}B \cong W_{D_6} \backslash W / W_{D_6}$ where W is the Weyl group of type E_7 and W_{D_6} is the parabolic subgroup of W of type D_6. Using Magma it is easy to compute the double cosets $W_{D_6} \backslash W / W_{D_6}$ and it follows that there are exactly five of them. Furthermore, the coset representatives corresponding to these 5 cosets, let us say $r_1, \dots, r_5$, are all involutions, hence for all $i = 1, \dots 5$ we have $r_i^{-1} \in W_{D_6} r_i W_{D_6}$. Through the abovementioned isomorphism and using the bijective correspondence between the G-orbitals on $Y \times Y$ and the double cosets of G with respect to K, we can see this to be equivalent to the fact that the orbitals on $Y \times Y$ are all self paired. But this implies that the permutation character consists of 5 irreducible characters, hence it is multiplicity free (see [5, p. 63]). This completes the proof. □

We can say a little bit more about these 5 orbits. Note that since $|K \backslash G / K| = |K \backslash N / K|$ the permutation rank of ρ is equal to the number of K-orbits on the set $\{ Kn \mid n \in N \}$, but this set can be identified with the set $Y' := \{ \mathbb{F}_q x_{\alpha_0}^n \mid n \in N \}$. For each element $x_\beta \in Y'$, the inner product $\langle \alpha_0, \beta \rangle$ returns a value in $\{-2, -1, 0, 1, 2\}$. This is an invariant for the action of K on these pairs of root elements, and since K stabilizes $\mathbb{F}_q x_{\alpha_0}$ it is also an invariant for the action of K on the set Y' of root elements. Hence the values $\{-2, -1, 0, 1, 2\}$ correspond to the five distinct orbits.

It follows from the theorem that there are 5 irreducible characters $\rho_1, \dots, \rho_5$ such that $\rho = \sum_{i=1}^{5} \rho_i$. We extend $\rho_1, \dots, \rho_5$ to an orthonormal basis of irreducible characters $\rho_1, \dots, \rho_k$ for the space of class functions and write $\pi = \sum_{i=1}^{k} c_i \rho_i$ for some constants $c_i \in \mathbb{N}$. The inner product $\langle \pi, \rho \rangle$ can be rewritten as $\langle \pi, \rho \rangle = \sum_{i=1}^{k} \sum_{j=1}^{5} c_i \langle \rho_i, \rho_j \rangle = \sum_{i=1}^{5} c_i$. On the other hand, by applying the theory of Lemma 2.7 and Frobenius reciprocity, we see that $\langle \pi, \rho \rangle = \langle (1_H)^G, \rho \rangle = \langle 1_H, \rho|_H \rangle = n$ where n equals the number of H-orbits on Y. If $n > 5$ then $c_i > 1$ for some $i \in \{1, \dots, 5\}$ and π is not multiplicity free. It follows from the next claim that this is indeed the case. next claim.

Claim 5.1 *There are at least 6 H-orbits on Y.*

Proof: Note that the function DC($\cdot$) as defined in appendix B.2 is an invariant on the H-orbits of Y. Thus the number of different values it takes on Y is a lower bound on the number of H orbits on Y. Using Magma we found that $\text{DC}(Y) \supseteq \{30, 31, 34, 39, 46, 49\}$, proving the claim. □

Element $y \in Y$	DC(y)
$x_{\alpha_2} + x_{-\alpha_2}$	30
$x_{-\alpha_{53}} + x_{\alpha_{27}} + x_{\alpha_{17}} + x_{-\alpha_4} + x_{\alpha_{50}} + x_{-\alpha_{33}} + x_{-\alpha_{22}} + x_{-\alpha_9}$	31
$x_{-\alpha_{61}} + x_{\alpha_1} + x_{-\alpha_{49}} + x_{-\alpha_{28}}$	34
$x_{\alpha_0} + x_{-\alpha_{59}}$	39
x_{α_2}	46
x_{α_0}	49

Table 2: 6 different H-orbits on Y exposed

We now state the main theorem of this thesis.

Theorem 5.3 *Let G be the Chevalley group of Lie type E_7 and of adjoint isogeny type over a field* $\mathrm{GF}(q)$ *of characteristic 2. Let H be the subgroup of G of Lie type A_7. Finally, let $-x_{\alpha_0}$ be the highest root of L_{E_7} of adjoint isogeny type and define $Y := \{ \mathbb{F}_q x_{\alpha_0}^g \mid g \in G \}$.*
Then there are exactly 7 H-orbits on Y.

The proof of this theorem will cover the remainder of this thesis and consists of a few parts. In section 6 we introduce a direct sum decomposition of L_{E_7} into two H-modules. In section 7 we construct a nice representation of these H-modules, for which we can associate H with the special linear group. In sections 8 and 9 we set up the theory to compute the stabilizers in H of the elements from each of these two submodules. Finally, in section 10 we use the direct sum decomposition to compute the stabilizers in H of 7 explicit elements from Y, each having a different stabilizer, thus each in a different H-orbit on Y. In section 11 information on each of the 7 H-orbits is summarized. Since the sizes of the 7 H-orbits sum up to the size of Y, we conclude that there are indeed no other orbits then these seven.

Given the contents of this section, the following Corollary follows from Claim 5.1 or Theorem 5.3.

Corollary 5.1 *There does not exist a distance-transitive graph with automorphism group $E_7(q)$ and vertex stabilizer $A_7(q) \cdot 2$ over a field* $\mathrm{GF}(q)$ *of characteristic* 2.

6 An $A_7(q) \cdot 2$-decomposition of L_{E_7} into $L_{A_7} \oplus W$

Since we want to find the H orbits on Y and Y is a subset of L_{E_7}, we are interested in the action of H on L_{E_7}. We will show that L_{E_7} decomposes into a direct sum of two H-modules. Hence the study of the H-action on L_{E_7} can be split up into smaller studies of the H-action on each of the two summands.

All roots in Φ (and thus also those in Ψ) can be expressed as a sum of fundamental roots $\alpha_1, \dots, \alpha_7 \in \Phi$. For each $i = 1, \dots, 7$ we define $c_i : \Phi \to \{\pm 0, \pm 1, \pm 2, \pm 3\}$ as the function taking a root $\alpha \in \Phi$ to the coefficient of the fundamental root α_i in α. The roots in Ψ can be characterized by looking at their second coordinate. Clearly

$$\Psi = \{ \alpha \in \Phi \mid c_2(\alpha) = 0 \bmod 2 \}$$

and

$$\Phi \setminus \Psi = \{ \alpha \in \Phi \mid c_2(\alpha) = 1 \bmod 2 \}.$$

Now it is easy to prove the following lemma:

Lemma 6.1 *$L_{E_7} = L_{A_7} \oplus W$, where L_{A_7} is the Lie algebra of type A_7 embedded in L_{E_7} and W is a 70 dimensional H-module.*

Proof: Remember that L_{A_7} is the subalgebra of L_{E_7} generated by the root elements $x_\alpha, (\alpha \in \Psi)$. We define $W := \langle x_\alpha \in L_{E_7} \mid \alpha \in \Phi \setminus \Psi \rangle$. The action of H on L_{A_7} and on W can be described by the action of its generators on the root elements of L_{A_7} and W respectively.

- The action of σ_δ on the root elements $x_{\alpha_0}, \dots, x_{\alpha_7}$ is given by $x_{\alpha_i}^{\sigma_\delta} = x_{\alpha_i \delta}$. It is clear that $c_2(\alpha_i) = c_2(\alpha_i \delta)$ for all $i = 1, \dots, 7$. Since σ_δ permutes the root elements and all roots are linear combinations of the fundamental roots we conclude that $x_\beta^{\sigma_\delta} \in L_{A_7}$ if $x_\beta \in L_{A_7}$ and $x_\beta^{\sigma_\delta} \in W$ if $x_\beta \in W$.
- Let $X_\alpha(t) \in H$. Since $\alpha \in \Psi$, this implies that $c_2(\alpha) = 0$. Let x_β be an arbitrary root element of L_{E_7}. Then $x_\beta^{X_\alpha(t)}$ is a linear combination of terms $x_{i\alpha+\beta}$ where $i = 0, 1, 2, 3$ (not all terms necessarily occur). Now $c_2(i\alpha + \beta) = ic_2(\alpha) + c_2(\beta) = c_2(\beta)$ thus if $x_\beta \in L_{A_7}$ then $x_\beta^{X_\alpha(t)} \in L_{A_7}$ and if $x_\beta \in W$ then $x_\beta^{X_\alpha(t)} \in W$.

Thus $L_{A_7}^H \subseteq L_{A_7}$ and $W^H \subseteq W$ and $L_{E_7} = L_{A_7} \oplus W$ as H-modules. □

7 The use of matrix representations

We wish to study the H-action on $L_{E_7} = L_{A_7} \oplus W$ (as H-modules). In this section we translate this into the study of the $\mathrm{SL}_8(q) \cdot 2$-action on $\mathrm{sl}_8(q) \oplus \Lambda^4 V$ (as $\mathrm{SL}_8(q) \cdot 2$-modules) where V is the natural $\mathrm{SL}_8(q)$ module.

7.1 An explicit identification between L_{A_7} and $\mathrm{sl}_8(q)$

Let $M_{d,d}(q)$ denote the matrix algebra of all $d \times d$ matrices over $\mathrm{GF}(q)$. We introduce *elementary* matrices $E_{ij} \in M_{d,d}(q)$ for $i, j = 1, \dots, d$ $(i \neq j)$ which have an entry 1 on position (i, j) and are zero on all other positions. Thus

$$(E_{ij})_{st} = \delta_{is}\delta_{jt}, \qquad s, t = 1, \dots, d. \tag{1}$$

where $(E_{ij})_{st}$ describes the element on the sth row and tth column of the matrix E_{ij}. The δ_{ij} denotes the Kronecker delta, returning the value 1 if i is equal to j and the value 0 otherwise.

Recall that $\mathrm{sl}_d(q)$ is the algebra of all $d \times d$ matrices over $\mathrm{GF}(q)$ with trace zero, equipped with the Lie bracket ($[M, N] = MN - NM$ for all $M, N \in \mathrm{sl}_d(q)$). It is well known (see e.g. [7]) that $\mathrm{sl}_d(q)$ is a Lie algebra of type A_{d-1}.

We are interested in the Lie algebra $\mathrm{sl}_8(q)$. A Chevalley basis for $\mathrm{sl}_8(q)$ is given by

$$\{\, E_{ij} \mid 1 \leq i, j \leq 8, i \neq j \,\} \cup \{\, H_{i,i+1} \mid 1 \leq i \leq 7 \,\},$$

where the matrices $H_{i,i+1}$ are defined by $H_{i,j} := [E_{ij}, E_{ji}]$. They generate the Cartan subalgebra, which consists of all diagonal matrices of trace zero.

The Lie algebra $\mathrm{sl}_8(q)$ is isomorphic to L_{A_7} and the isomorphism $\phi\colon L_{A_7} \to \mathrm{sl}_8(q)$ maps the generating elements of L_{A_7} in the following way:

$$\begin{aligned}
\phi(x_{\alpha_0}) &= E_{1,2}, & \phi(x_{-\alpha_0}) &= E_{2,1},\\
\phi(x_{\alpha_1}) &= E_{2,3}, & \phi(x_{-\alpha_1}) &= E_{3,2},\\
\phi(x_{\alpha_3}) &= E_{3,4}, & \phi(x_{-\alpha_3}) &= E_{4,3},\\
\phi(x_{\alpha_4}) &= E_{4,5}, & \phi(x_{-\alpha_4}) &= E_{5,4},\\
\phi(x_{\alpha_5}) &= E_{5,6}, & \phi(x_{-\alpha_5}) &= E_{6,5},\\
\phi(x_{\alpha_6}) &= E_{6,7}, & \phi(x_{-\alpha_6}) &= E_{7,6},\\
\phi(x_{\alpha_7}) &= E_{7,8}, & \phi(x_{-\alpha_7}) &= E_{8,7}.
\end{aligned}$$

Using the fact that ϕ is invariant under the Lie bracket, this determines ϕ uniquely. One readily checks that the positive root elements all correspond to elementary upper diagonal matrices and the negative root elements all correspond to elementary lower diagonal matrices. This isomorphism is also made explicit in for example [7].

7.2 An explicit identification between $A_7(q) \cdot 2$ and $\mathrm{SL}_8(q) \cdot 2$

The set-up of $\mathrm{sl}_8(q)$ also determines an explicit isomorphism $\tilde{\phi}$ between its automorphism group $\mathrm{SL}_8(q)$ and the group $A_7(q)$ of all automorphisms of L_{A_7}. Recall that the group $A_7(q)$ is generated by the elements $X_\alpha(t)$ for $\alpha \in \Psi$ and $t \in \mathbb{F}_q$. Now since ϕ respects the Lie bracket, $\mathrm{SL}_8(q)$ is generated by the elements

$$\tilde{\phi}(X_\alpha(t)) = \tilde{\phi}(\exp(t\mathrm{ad}_{x_\alpha})) = \exp(t\mathrm{ad}_{\phi(x_\alpha)})$$

for $\alpha \in \Psi$ and $t \in \mathbb{F}_q$. Note that all elementary matrices $\phi(x_\alpha)$ are nilpotent. Furthermore, we have $(\phi(x_\alpha))^2 = 0$ for all elementary matrices. Thus

$$\begin{aligned}\tilde{\phi}(X_\alpha(t))(M) &= \exp(t\mathrm{ad}_{\phi(x_\alpha)})(M) \\ &= (\exp(t\phi(x_\alpha)))^{-1}M(\exp(t\phi(x_\alpha))) \\ &= (I + t\phi(x_\alpha))^{-1}M(I + t\phi(x_\alpha)) \\ &= M^{(I+t\phi(x_\alpha))}\end{aligned}$$

for all $M \in \mathrm{sl}_8(q)$. We also write $\tilde{X}_\alpha(t) := I + t\phi(x_\alpha)$. Thus $\mathrm{SL}_8(q)$ is generated by the elements $\tilde{X}_\alpha(t)$, corresponding to the generators $X_\alpha(t)$ of $A_7(q)$, and it acts on $\mathrm{sl}_8(q)$ by conjugation. This action is called the *adjoint action*.

We now extend $\tilde{\phi}$ to an isomorphism of H to $\mathrm{SL}_8(q)\cdot 2$. We define $\varsigma := \tilde{\phi}(w_0^\Phi)$. From the simple form of the elementary matrices, it follows immediately that $\varsigma\colon M \mapsto M^T$ for all $M \in \mathrm{sl}_8(q)$. We write $\tilde{H} := \mathrm{SL}_8(q)\cdot < \varsigma >$. Then $\tilde{\phi}$ establishes an isomorphism

$$H \cong \tilde{H}. \tag{2}$$

Thus we can interpret $\mathrm{sl}_8(q)$ as an H-module through the above identification.

Let $\Pi := \Pi_\Psi$. Remember that $\Pi = \{\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. Now let $\tilde{n}_\alpha$, $\alpha \in \Pi$ be the fundamental reflections in $\mathrm{SL}_8(q)$ corresponding to the fundamental reflections $\dot{n}_\alpha$, $\alpha \in \Pi$ in $A_7(q)$. Then

$$\tilde{n}_\alpha = \tilde{X}_\alpha(1)\tilde{X}_{-\alpha}(-1)\tilde{X}_\alpha(1) \text{ for } \alpha \in \Pi_\Psi$$

and we compute $\tilde{n}_\alpha$, $\alpha \in \Pi$ explicitly:

$$\tilde{n}_{\alpha_0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ldots, \tilde{n}_{\alpha_7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

7.3 A simultaneous identification between W and $\Lambda^4 V$

Let V be the natural $\mathrm{SL}_8(q)$ module with a natural basis $v_1, \ldots, v_8$. Then $\Lambda^4 V$ is the *fourth exterior power* of V. It has a basis given by $\{ v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \mid 1 \le i_1 < i_2 < i_3 < i_4 \le 8 \}$. Since we work over a field of characteristic 2, the wedge product is independent of the order of the factors, thus $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} = v_{i_1^\sigma} \wedge v_{i_2^\sigma} \wedge v_{i_3^\sigma} \wedge v_{i_4^\sigma}$ for all $\sigma \in \mathrm{Sym}(\{i_1, i_2, i_3, i_4\})$ for $1 \le i_1, i_2, i_3, i_4 \le 8$.

The right action of $\mathrm{SL}_8(q)$ on V induces a natural action of $\mathrm{SL}_8(q)$ on $\Lambda^4 V$. For $M \in \mathrm{SL}_8(q)$ and $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \in \Lambda^4 V$, the action is determined by

$$(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4})^M = v_{i_1}^M \wedge v_{i_2}^M \wedge v_{i_3}^M \wedge v_{i_4}^M.$$

This action is called the *alternating action*[6]. Note that both W and $\Lambda^4 V$ are 70 dimensional over the finite field $\mathrm{GF}(q)$. We determine an explicit isomorphism θ of H-modules between W and $\Lambda^4 V$. We use the symbol $\sim$ to denote that two elements are isomorphic (through the isomorphism θ). Let $x_{\alpha_2} \sim v_1 \wedge v_2 \wedge v_3 \wedge v_4$. For all $\beta \in \Phi \setminus \Psi$ there exist fundamental reflections w_i, $i = 1, \ldots, k$ such that $\beta = w_1 \ldots w_k(\alpha_2)$. According to Lemma 2.5, this implies that there exist fundamental reflections $n_1, \ldots, n_k$ such that $x_\beta = x_{\alpha_2}^{n_k \cdots n_1}$. We write $n_\beta := n_k \ldots n_1$. We now use the action of N on the root elements to construct the isomorphism θ.

[6]Since the elements in $\mathrm{SL}_8(q)$ simply act on the elements $v \in V$ through right multiplication by a matrix, we also use the normal script notation vM for the image of v under the action of $M \in \mathrm{SL}_8(q)$.

For all $\beta \in \Phi \setminus \Psi$ we have

$$\theta(x_\beta) = \theta(x_{\alpha_2}^{n_\beta}) = (v_1 \wedge v_2 \wedge v_3 \wedge v_4)^{\tilde{n}_\beta},$$

establishing the isomorphism between the generators of W and the generators of $\Lambda^4 V$. By construction, the group N acts on W in the same way that $\tilde{\phi}(N)$ acts on $\Lambda^4 V$. The action of ς on $\Lambda^4 V$ is easily seen to be given by

$$\varsigma\colon v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \mapsto v_{i_5} \wedge v_{i_6} \wedge v_{i_7} \wedge v_{i_8}$$

for all distinct $i_1, \ldots i_8$ such that $1 \leq i_1, \ldots, i_8 \leq 8$. Thus the group H acts on W in the same way that $\tilde{H}$ acts on $\Lambda^4 V$. Thus we can interpret $\Lambda^4 V$ as an H-module through the identification in (2).

Lemma 7.1 *There exists an isomorphism γ of H-modules between $L_{A_7} \oplus W$ and $\mathrm{sl}_8(q) \oplus \Lambda^4 V$.*

Proof: Define the morphism γ from $L_{A_7} \oplus W$ to $\mathrm{sl}_8(q) \oplus \Lambda^4 V$ by its action on the generators:

$$\gamma(x_\alpha) = \begin{cases} \phi(x_\alpha) & \text{if } \alpha \in \Psi \\ \theta(x_\alpha) & \text{if } \alpha \in \Phi \setminus \Psi \end{cases}$$

This morphism γ is clearly bijective since both ϕ and θ are bijective. Both $\mathrm{sl}_8(q)$ and $\Lambda^4 V$ can be interpreted as H-modules through the same identification in (2) hence γ is an isomorphism of H-modules. □

Thus determining the H-orbits on L_{E_7} is equivalent to determining the $\tilde{H}$-orbits on $\mathrm{sl}_8(q) \oplus \Lambda^4 V$. Using the latter representation, in which $\tilde{H}$ is almost[7] a matrix group, has the benefit that it allows for easy and insightful calculations.

8 The stabilizer in $\mathrm{SL}_d(q)$ of the adjoint action

The group $\mathrm{SL}_d(q)$ acts on $\mathrm{sl}_d(q)$ by conjugation. In the next subsection we look at the stabilizer in $\mathrm{SL}_d(q)$ of a matrix E_{ij}. After that we determine the stabilizer in $\mathrm{SL}_d(q)$ of a more general matrix $E_{i_1 j_1} + \ldots + E_{i_n j_n}$ where all i_k and j_k are different and $n \geq 1$. Finally we determine the stabilizer of the one dimensional vector space, spanned by such a matrix.

8.1 The stabilizer of E_{ij}

Let E_{ij} be as in (1) for arbitrary but fixed $i, j = 1, \ldots, d$ ($i \neq j$). Since $\mathrm{SL}_d(q)$ acts on $\mathrm{sl}_d(q)$ by conjugation, its stabilizer in $\mathrm{SL}_d(q)$ is given by

$$\mathrm{Stab}_{\mathrm{SL}_d(q)}(E_{ij}) = \{\, M \in \mathrm{SL}_d(q) \mid ME_{ij} = E_{ij}M \,\}.$$

Now suppose $M \in \mathrm{Stab}_{\mathrm{SL}_d(q)}(E_{ij})$. Then in order for the two matrices ME_{ij} and $E_{ij}M$ to be equal, they must be equal entrywise. Thus

$$(ME_{ij})_{st} = (E_{ij}M)_{st}$$

must hold for all $s, t \in \{1, \ldots, d\}$. Since E_{ij} is very sparse we can simplify both sides;

$$\begin{aligned} (ME_{ij})_{st} &= \sum_{u=1}^{8} (M)_{su}(E_{ij})_{ut} = (M)_{si}(E_{ij})_{it} = (M)_{si}\delta_{tj} \\ (E_{ij}M)_{st} &= \sum_{u=1}^{8} (E_{ij})_{su}(M)_{ut} = (E_{ij})_{sj}(M)_{jt} = (M)_{jt}\delta_{is} \end{aligned}$$

[7]The involution ς on top, cannot be represented by a matrix.

from which we obtain the simple constraint $(M)_{si}\delta_{jt} = (M)_{jt}\delta_{is}$. Looking at the possible values of s and t this gives the following conditions on the stabilizing element $M \in \mathrm{SL}_d(q)$:

- $M_{ii} = M_{jj}$;
- $M_{jt} = 0$ for all $t \neq j$;
- $M_{si} = 0$ for all $s \neq i$.

Since the elements E_{ij} for $i, j \in \{1, \ldots, d\}, (i \neq j)$ are all conjugate in $\mathrm{SL}_d(q)$, we can speak of a *standard form* for the stabilizer in $\mathrm{SL}_d(q)$ of an element E_{ij}. We let this standard form be the stabilizer of the element $E_{1,2}$ which is the subgroup of $\mathrm{SL}_d(q)$ consisting of matrices of the form

$$M = \begin{pmatrix} \lambda & * & * & * & \cdots & * \\ 0 & \lambda & 0 & 0 & \cdots & 0 \\ 0 & * & & & & \\ 0 & * & & & & \\ \vdots & \vdots & & & \mu M' & \\ 0 & * & & & & \end{pmatrix}, \tag{3}$$

where $\lambda, \mu \in \mathbb{F}_q^\times$ such that $\lambda^2\mu^{d-2} = 1$, the $*$-symbol denotes arbitrary values and the $(d-2) \times (d-2)$ matrix $M' \in \mathrm{SL}_{d-2}(q)$. This follows since $\det(M) = \lambda^2\mu^{d-2}\det(M') = \det(M')$.

8.2 The stabilizer of $E_{i_1j_1} + \ldots + E_{i_nj_n}$

We now describe the stabilizer of an element $\overline{v} := E_{i_1j_1} + \ldots + E_{i_nj_n} \in \mathrm{sl}_d(q)$ for distinct i_k, j_l, where $k, l = 1, \ldots, n$. Note that $2n \leq d$ since all the indices must be different. Assume that $M \in \mathrm{SL}_d(q)$ is the stabilizer of $\overline{v}$. Then $M\overline{v} = \overline{v}M$ and similarly to the computations in the last subsection we obtain

$$(M)_{si_1}\delta_{tj_1} + (M)_{si_2}\delta_{tj_2} + \ldots + (M)_{si_n}\delta_{tj_n} = (M)_{j_1t}\delta_{si_1} + (M)_{j_2t}\delta_{si_2} + \ldots + (M)_{j_nt}\delta_{si_n}$$

The conditions on M can be summarized as follows

- $M_{j_kt} = 0$ for all $k \in \{1, \ldots, n\}$, and $t \neq j_l$ for all $l \in \{1, \ldots, n\}$; (4)
- $M_{si_k} = 0$ for all $k \in \{1, \ldots, n\}$, and $s \neq i_l$ for all $l \in \{1, \ldots, n\}$; (5)
- $M_{i_ki_l} = M_{j_kj_l}$ for all $k, l \in \{1, \ldots, n\}$. (6)

In the previous subsection we introduced the notion of a standard form for the stabilizer of an element E_{ij}, being the stabilizer of the element $E_{1,2}$. (See (3).) In a similar way we now introduce a standard form for the stabilizer of a general element $E_{i_1j_1} + \ldots + E_{i_nj_n}$.

Lemma 8.1 *Let $\overline{v} := E_{1,2n} + E_{2,2n-1} + \ldots + E_{n,n+1}$ for some n, $2 \leq 2n \leq d$. Then*

$$Stab_{\mathrm{SL}_d(q)}(\overline{v}) = \left\{ \begin{pmatrix} \lambda A & *_{n,n} & *_{n,d-2n} \\ 0_{n,n} & \lambda\widehat{A} & 0_{n,d-2n} \\ 0_{d-2n,n} & *_{d-2n,n} & \mu M' \end{pmatrix} \middle| \begin{array}{l} A \in SL_n(q), M' \in SL_{d-2n}(q), \\ \lambda, \mu \in \mathbb{F}_q^\times, \mu^{d-2n} = \lambda^{-2n} \end{array} \right\}, \tag{7}$$

where $0_{i,j}$ denotes the $i \times j$ zero matrix and $*_{ij}$ denotes an arbitrary $i \times j$ matrix over $\mathrm{GF}(q)$. *The matrix $\widehat{A}$ is determined uniquely by the matrix A and is defined by*

$$\widehat{A} = NAN^{-1} \qquad \text{where } N = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & 1 & \iddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}$$

Note that the matrix $\widehat{A}$ can be obtained from the matrix A by rotating it over 180 degrees.

Proof: The general form of the first n columns of a stabilizing element follows from condition (4). Moreover $\lambda A \in GL_n(q)$ since the matrix has to have determinant unequal to zero. This is equivalent to the condition that $\lambda \in \mathbb{F}_q^\times$ and $A \in SL_n(q)$. The general form of rows $n+1$ up to $2n$ of this matrix now follows directly from conditions (5) and (6). For each $\lambda \in \mathbb{F}_q^\times$ there is a unique[8] element $\mu \in \mathbb{F}_q$ such that the determinant of the total matrix $\lambda^{2n}\mu^{d-2n} = 1$. Therefore the sub matrix in the lower right corner has to have determinant μ^{d-2n}. Hence the stabilizing matrices have the above form. □

8.3 The stabilizer of the one dimensional vector space

In the specific cases that we study, we are not directly interested in the stabilizer of the mentioned elements, but in the stabilizer of one dimensional vector spaces spanned by these elements. From the above lemma we easily obtain the form of such a stabilizer.

Lemma 8.2 *Let $\overline{v}$ be as in Lemma 8.1. Then the stabilizer of $\langle\overline{v}\rangle$ in $\mathrm{SL}_d(q)$ is given by*

$$Stab_{\mathrm{SL}_d(q)}(\langle\overline{v}\rangle) = \left\{ \begin{pmatrix} \lambda A & *_{n,n} & *_{n,d-2n} \\ 0_{n,n} & \mu\widehat{A} & 0_{n,d-2n} \\ 0_{d-2n,n} & *_{d-2n,n} & \nu M' \end{pmatrix} \middle| \begin{array}{l} A \in SL_n(q), M' \in SL_{d-2n}(q), \\ \lambda, \mu, \nu \in \mathbb{F}_q^\times, \nu^{d-2n} = \lambda^{-n}\mu^{-n} \end{array} \right\}, \quad (8)$$

where $\widehat{A}$ is defined as in Lemma 8.1.

Proof: The matrices in $\mathrm{Stab}_{\mathrm{SL}_d(q)}(\langle\overline{v}\rangle)$ map $\overline{v}$ onto a scalar multiple $\xi \in \mathbb{F}_q$ of itself. Thus for an element $M \in \mathrm{Stab}_{\mathrm{SL}_d(q)}(\langle\overline{v}\rangle)$ it must hold that $M\overline{v} = \xi\overline{v}M$ for some $\xi \in \mathbb{F}_q$. One easily sees that condition (6) changes to

$$M_{i_k i_l} = \xi M_{j_k j_l} \text{ for all } k, l \in \{1, \ldots, n\},$$

introducing an extra degree of freedom into the matrix M. Note however, that since M needs to be an invertible matrix we actually need $\xi \in \mathbb{F}_q^\times$, for if $\xi = 0$ then for each $k = 1, \ldots, n$, the i_k'th column of M would consist of zeros only, and M would not be invertible. □

9 The stabilizer in $\mathrm{SL}_d(q)$ of the alternating action

Let V be the natural module of $\mathrm{SL}_d(q)$. This action induces a right action of $\mathrm{SL}_d(q)$ on $\Lambda^4 V$ in a natural way. In the next subsection, we investigate the stabilizer of an arbitrary basis element $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \in \Lambda^4 V$ (where the indices $i_1, \ldots, i_4$ are all different). After that we determine the stabilizer of sums of basis elements.

9.1 The stabilizer of $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4}$

Let $\underline{v} := v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \in \Lambda^4(V)$ for arbitrary but fixed indices $i_1, \ldots, i_4$ (all different). Since $\mathrm{SL}_d(q)$ acts on $\Lambda^4 V$ by right action, its stabilizer in $\mathrm{SL}_d(q)$ is given by

$$\mathrm{Stab}_{\mathrm{SL}_d(q)}(\underline{v}) = \{\, M \in \mathrm{SL}_d(q) \mid \underline{v}\, M = \underline{v} \,\} \quad (9)$$

[8]This element is unique up to a factor $\sqrt[d-2n]{1}$.

Suppose that $M = (m_{kl})_{1\leq k,l\leq 8} \in \mathrm{SL}_d(q)$ stabilizes $\underline{v}$. Then $\underline{v}\, M = \underline{v}$. Writing this out, we see that

$$\begin{aligned}
\underline{v}\, M &= (v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4})M \\
&= v_{i_1}M \wedge v_{i_2}M \wedge v_{i_3}M \wedge v_{i_4}M \\
&= (\sum_{k_1=1}^{d} m_{i_1k_1}v_{k_1}) \wedge (\sum_{k_2=1}^{d} m_{i_2k_2}v_{k_2}) \wedge (\sum_{k_3=1}^{d} m_{i_3k_3}v_{k_3}) \wedge (\sum_{k_4=1}^{d} m_{i_4k_4}v_{k_4}) \\
&= \sum_{k_1=1}^{d}\sum_{k_2=1}^{d}\sum_{k_3=1}^{d}\sum_{k_4=1}^{d} (m_{i_1k_1}m_{i_2k_2}m_{i_3k_3}m_{i_4k_4})\, v_{k_1} \wedge v_{k_2} \wedge v_{k_3} \wedge v_{k_4}
\end{aligned}$$

Let $\mathcal{K}$ denote the set of all $\binom{d}{4}$ combinations of 4 elements from the set $\{1, \ldots, d\}$. Since we work over a field of characteristic 2, we can rewrite the above sum further;

$$\underline{v}\, M = \sum_{\{k_1,k_2,k_3,k_4\}\in\mathcal{K}} \begin{vmatrix} m_{i_1k_1} & m_{i_1k_2} & m_{i_1k_3} & m_{i_1k_4} \\ m_{i_2k_1} & m_{i_2k_2} & m_{i_2k_3} & m_{i_2k_4} \\ m_{i_3k_1} & m_{i_3k_2} & m_{i_3k_3} & m_{i_3k_4} \\ m_{i_4k_1} & m_{i_4k_2} & m_{i_4k_3} & m_{i_4k_4} \end{vmatrix} v_{k_1} \wedge v_{k_2} \wedge v_{k_3} \wedge v_{k_4} \tag{10}$$

We have used here that we are working over $\mathrm{GF}(q)$ which is a field of characteristic 2. Hence there is no difference between addition and subtraction.

Now since the condition is that $\underline{v}\, M = \underline{v}$, from (10) we obtain $\binom{d}{4}$ different equations, one for each basis element. Thus

$$\begin{vmatrix} m_{i_1k_1} & m_{i_1k_2} & m_{i_1k_3} & m_{i_1k_4} \\ m_{i_2k_1} & m_{i_2k_2} & m_{i_2k_3} & m_{i_2k_4} \\ m_{i_3k_1} & m_{i_3k_2} & m_{i_3k_3} & m_{i_3k_4} \\ m_{i_4k_1} & m_{i_4k_2} & m_{i_4k_3} & m_{i_4k_4} \end{vmatrix} = \begin{cases} 1 & \text{if } v_{k_1} \wedge v_{k_2} \wedge v_{k_3} \wedge v_{k_4} = v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

This means that the columns $i_1, \ldots, i_4$ of M limited to the rows $i_1, \ldots, i_4$ are linearly independent. Moreover, all other 4-tuples of columns, limited to these rows, are linearly dependent. From this it follows that column $i_k, k \notin \{1, \ldots, 4\}$, limited to rows $i_1, \ldots, i_4$, is all zero, since this column is dependent on every 3-tuple of the linear independent columns $i_1, \ldots, i_4$. As a *standard form* of M we consider the stabilizer of the first basis element of $\Lambda^4 V$, $v_1 \wedge v_2 \wedge v_3 \wedge v_4$. This gives us the standard form

$$M = \begin{pmatrix} M' & 0_{4,d-4} \\ *_{d-4,4} & M'' \end{pmatrix}, \tag{12}$$

where $M' \in \mathrm{SL}_4(q)$ and $M'' \in \mathrm{SL}_{d-4}(q)$.

9.2 The stabilizer of $v_{i_{1\,1}} \wedge v_{i_{1\,2}} \wedge v_{i_{1\,3}} \wedge v_{i_{1\,4}} + \ldots + v_{i_{n\,1}} \wedge v_{i_{n\,2}} \wedge v_{i_{n\,3}} \wedge v_{i_{n\,4}}$

In this subsection we will look at the stabilizer in $\mathrm{SL}_d(q)$ of a general element of $\Lambda^4 V$. Let $\underline{v} = v_{i_{1\,1}} \wedge v_{i_{1\,2}} \wedge v_{i_{1\,3}} \wedge v_{i_{1\,4}} + \ldots + v_{i_{n\,1}} \wedge v_{i_{n\,2}} \wedge v_{i_{n\,3}} \wedge v_{i_{n\,4}}$ be an arbitrary but fixed element from $\Lambda^4 V$. Suppose that $M = (m_{kl})_{1\leq k,l\leq d} \in \mathrm{SL}_d(q)$ stabilizes $\underline{v}$. In a similar way to the computations in the previous subsection, we obtain an equation similar to (10):

$$\underline{v}\, M = \sum_{\{k_1,k_2,k_3,k_4\}\in\mathcal{K}} \sum_{j=1}^{n} \begin{vmatrix} m_{i_{j\,1}k_1} & m_{i_{j\,1}k_2} & m_{i_{j\,1}k_3} & m_{i_{j\,1}k_4} \\ m_{i_{j\,2}k_1} & m_{i_{j\,2}k_2} & m_{i_{j\,2}k_3} & m_{i_{j\,2}k_4} \\ m_{i_{j\,3}k_1} & m_{i_{j\,3}k_2} & m_{i_{j\,3}k_3} & m_{i_{j\,3}k_4} \\ m_{i_{j\,4}k_1} & m_{i_{j\,4}k_2} & m_{i_{j\,4}k_3} & m_{i_{j\,4}k_4} \end{vmatrix} v_{k_1} \wedge v_{k_2} \wedge v_{k_3} \wedge v_{k_4} \tag{13}$$

This again gives us one condition for each basis element;

$$\sum_{j=1}^{n} \begin{vmatrix} m_{i_{j\,1}k_1} & m_{i_{j\,1}k_2} & m_{i_{j\,1}k_3} & m_{i_{j\,1}k_4} \\ m_{i_{j\,2}k_1} & m_{i_{j\,2}k_2} & m_{i_{j\,2}k_3} & m_{i_{j\,2}k_4} \\ m_{i_{j\,3}k_1} & m_{i_{j\,3}k_2} & m_{i_{j\,3}k_3} & m_{i_{j\,3}k_4} \\ m_{i_{j\,4}k_1} & m_{i_{j\,4}k_2} & m_{i_{j\,4}k_3} & m_{i_{j\,4}k_4} \end{vmatrix} = \begin{cases} 1 & \text{if } v_{k_1} \wedge v_{k_2} \wedge v_{k_3} \wedge v_{k_4} = \\ & v_{i_{p\,1}} \wedge v_{i_{p\,2}} \wedge v_{i_{p\,3}} \wedge v_{i_{p\,4}} \\ & \text{for some } p \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

We do not work out this formula for all possible cases. We merely derived it to be able to use it in specific cases we will encounter.

9.3 The stabilizer of the one dimensional vector space

Just as in the previous section we are not directly interested in the stabilizer of $\underline{v}$ itself, but in the stabilizer of the one dimensional vector space $\langle \underline{v} \rangle$. Since an element $M = (m_{kl})_{1 \le k,l \le d}$ from the stabilizer must now satisfy $\underline{v}M = \xi \underline{v}$ for some $\xi \in \mathbb{F}$, one easily sees that the conditions on M described in (14) now changes into

$$\sum_{j=1}^{n} \begin{vmatrix} m_{i_j\,1 k_1} & m_{i_j\,1 k_2} & m_{i_j\,1 k_3} & m_{i_j\,1 k_4} \\ m_{i_j\,2 k_1} & m_{i_j\,2 k_2} & m_{i_j\,2 k_3} & m_{i_j\,2 k_4} \\ m_{i_j\,3 k_1} & m_{i_j\,3 k_2} & m_{i_j\,3 k_3} & m_{i_j\,3 k_4} \\ m_{i_j\,4 k_1} & m_{i_j\,4 k_2} & m_{i_j\,4 k_3} & m_{i_j\,4 k_4} \end{vmatrix} = \begin{cases} \xi & \text{if } v_{k_1} \wedge v_{k_2} \wedge v_{k_3} \wedge v_{k_4} = \\ & v_{i_{p\,1}} \wedge v_{i_{p\,2}} \wedge v_{i_{p\,3}} \wedge v_{i_{p\,4}} \\ & \text{for some } p \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}, \tag{15}$$

where $\xi \in \mathbb{F}_q$.

In the simple case, where $\underline{v} = v_1 \wedge v_2 \wedge v_3 \wedge v_4$, it is easy to see that the stabilizer consists of matrices M of the form

$$M = \begin{pmatrix} \lambda M' & 0_{4,d-4} \\ *_{d-4,4} & \mu M'' \end{pmatrix} \tag{16}$$

such that $M' \in \mathrm{SL}_4(q), M'' \in \mathrm{SL}_{d-4}(q)$, and $\lambda, \mu \in \mathbb{F}_q^\times$ such that $\lambda^4 \mu^{d-4} = 1$.

10 Representatives from the 7 $A_7(q) \cdot 2$-orbits on the root elements of L_{E_7}

We wish to determine the 7 stabilizers under the action of $\tilde{H}$ of the one dimensional vector spaces spanned by specific elements $w_1, \ldots, w_7$, each isomorphic to an element from a different H-orbit on Y through the isomorphism γ, introduced in section 7.3.

Name	Element $y \in Y$	$\gamma(y)$
w_1	x_{α_0}	$E_{1,2}$
w_2	x_{α_2}	$v_1 \wedge v_2 \wedge v_3 \wedge v_4$
w_3	$x_{\alpha_0} + x_{-\alpha_{59}}$	$E_{1,2} + v_2 \wedge v_3 \wedge v_4 \wedge v_5$
w_4	$x_{-\alpha_{61}} + x_{\alpha_1} + x_{-\alpha_{49}} + x_{-\alpha_{28}}$	$E_{1,4} + E_{2,3} + v_3 \wedge v_4 \wedge v_5 \wedge v_6 + v_3 \wedge v_4 \wedge v_7 \wedge v_8$
w_5	$x_{-\alpha_{53}} + x_{\alpha_{27}} + x_{\alpha_{17}} + x_{-\alpha_4} +$	$v_1 \wedge v_5 \wedge v_6 \wedge v_7 + v_2 \wedge v_5 \wedge v_6 \wedge v_8 +$
	$x_{\alpha_{50}} + x_{-\alpha_{33}} + x_{-\alpha_{22}} + x_{-\alpha_9}$	$v_3 \wedge v_5 \wedge v_7 \wedge v_8 + v_4 \wedge v_6 \wedge v_7 \wedge v_8$
w_6	$x_{\alpha_2} + x_{-\alpha_2}$	$v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_5 \wedge v_6 \wedge v_7 \wedge v_8$
w_7	not explicitly known	$\xi v_1 \wedge v_2 \wedge v_3 \wedge v_4 + \xi^q v_5 \wedge v_6 \wedge v_7 \wedge v_8$, $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$

Table 3: The correspondence between elements from Y and elements from $\mathrm{sl}_8(q) \oplus \Lambda^4 V$

The fact that the union of these 7 H-orbits on Y, corresponding to the elements $w_1, \ldots, w_7$ forms Y, follows from computing the corresponding H-orbit sizes and showing that these sum up to the size of Y. Combining the results of the last two sections, we first determine the 7 stabilizers under the action of $\mathrm{SL}_8(q)$. In the end of this section we lift these stabilizers to stabilizers under the action of $\tilde{H}$.

- The element x_{α_0} corresponds to $w_1 = E_{1,2}$. From (8) we obtain

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_1 \rangle) = \left\{ \begin{pmatrix} \lambda & * & * & * & * & * & * & * \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{\nu M} \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{} \end{pmatrix} \;\middle|\; \begin{array}{l} M \in \mathrm{SL}_6(q), \\ \lambda, \mu, \nu \in \mathbb{F}_q^{\times}, \\ \nu^6 = \lambda^{-1}\mu^{-1} \end{array} \right\}.$$

  Thus

$$|\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_1 \rangle)| = (q-1)^2 q^6 q^7 |\mathrm{SL}_6(q)| = (q-1)^2 q^{28}(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1).$$

- The element x_{α_2} corresponds to $w_2 = v_1 \wedge v_2 \wedge v_3 \wedge v_4$. From (16) we obtain

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_2 \rangle) = \left\{ \begin{pmatrix} \lambda M & 0_{4,4} \\ *_{4,4} & \lambda^{-1} M' \end{pmatrix} \;\middle|\; \begin{array}{l} M, M' \in \mathrm{SL}_4(q) \\ \lambda \in \mathbb{F}_q^{\times} \end{array} \right\}.$$

  Consequently

$$|\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_2 \rangle)| = (q-1)q^{16}(|\mathrm{SL}_4(q)|)^2 = (q-1)q^{28}(q^2-1)^2(q^3-1)^2(q^4-1)^2.$$

- The element $x_{\alpha_0} + x_{-\alpha_{59}}$ corresponds to $w_3 = E_{12} + v_2 \wedge v_3 \wedge v_4 \wedge v_5$. From (8) we know that the stabilizer of $\langle E_{1,2} \rangle$ is

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle E_{1,2} \rangle) = \left\{ \begin{pmatrix} \lambda & * & * & * & * & * & * & * \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{\nu M} \\ 0 & * & \multicolumn{6}{c}{} \\ 0 & * & \multicolumn{6}{c}{} \end{pmatrix} \;\middle|\; \begin{array}{l} M \in \mathrm{SL}_6(q), \\ \lambda, \mu, \nu \in \mathbb{F}_q^{\times}, \\ \nu^6 = \lambda^{-1}\mu^{-1} \end{array} \right\}.$$

  Note that an element of this form maps $E_{1,2}$ to $\lambda^{-1}\mu E_{1,2}$ (this follows immediately by letting a matrix of this form act on $E_{1,2}$). Thus within this stabilizer we also want to map $v_2 \wedge v_3 \wedge v_4 \wedge v_5$ to $(\lambda^{-1}\mu) v_2 \wedge v_3 \wedge v_4 \wedge v_5$. Condition (15) restricts the structure of rows $2, 3, 4$, and 5. In particular the 4×4 submatrix

$$\begin{pmatrix} \mu & 0 & 0 & 0 \\ * & \nu m_{11} & \nu m_{12} & \nu m_{13} \\ * & \nu m_{21} & \nu m_{22} & \nu m_{23} \\ * & \nu m_{31} & \nu m_{32} & \nu m_{33} \end{pmatrix}$$

  of the above form, where m_{11} is the top left corner of the submatrix M, needs to have determinant $\lambda^{-1}\mu$. The other columns will consist of zeros. Thus it is easy to see that the stabilizer is given by

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_3 \rangle) = \left\{ \begin{pmatrix} \lambda & * & * & * & * & * & * & * \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & \multicolumn{3}{c}{} & 0 & 0 & 0 \\ 0 & * & \multicolumn{3}{c}{\nu M'} & 0 & 0 & 0 \\ 0 & * & \multicolumn{3}{c}{} & 0 & 0 & 0 \\ 0 & * & * & * & * & \multicolumn{3}{c}{} \\ 0 & * & * & * & * & \multicolumn{3}{c}{\xi M''} \\ 0 & * & * & * & * & \multicolumn{3}{c}{} \end{pmatrix} \;\middle|\; \begin{array}{l} M', M'' \in \mathrm{SL}_3(q), \\ \lambda, \mu, \nu, \xi \in \mathbb{F}_q^{\times}, \\ \nu^3 = \lambda\mu^{-2}, \xi^3 = \lambda^{-2}\mu \end{array} \right\}$$

Thus

$$|\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_3\rangle)| = (q-1)^2 q^9 (q^3)^4 q(|\mathrm{SL}_3(q)|)^2 = (q-1)^2 q^{28}(q^2-1)^2(q^3-1)^2.$$

- The element $x_{-\alpha_{61}} + x_{\alpha_1} + x_{-\alpha_{49}} + x_{-\alpha_{28}}$ corresponds to the element $w_4 = \overline{w_4} + \underline{w_4}$ where $\overline{w_4} = E_{14} + E_{23}$ and $\underline{w_4} = v_3 \wedge v_4 \wedge v_5 \wedge v_6 + v_3 \wedge v_4 \wedge v_7 \wedge v_8$. From (8) we know that the stabilizer of $\langle \overline{w_4} \rangle$ is

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle \overline{w_4}\rangle) = \left\{ \begin{pmatrix} \lambda M & *_{2,2} & *_{2,4} \\ 0_{2,2} & \mu \widehat{M} & 0_{2,4} \\ 0_{4,2} & *_{4,2} & \nu M' \end{pmatrix} \middle| \begin{array}{l} M \in \mathrm{SL}_2(q), M' \in \mathrm{SL}_4(q), \\ \lambda, \mu, \nu \in \mathbb{F}_q^\times, \nu^4 = \lambda^{-2}\mu^2 \end{array} \right\}.$$

The matrices of this form map $\overline{w_4}$ to $(\lambda^{-1}\mu)\overline{w_4}$ thus we look for those matrices in this stabilizer, that map $\underline{w_4}$ to $(\lambda^{-1}\mu)\underline{w_4}$.

We can write $\underline{w_4} = v_3 \wedge v_4 \wedge (v_5 \wedge v_6 + v_7 \wedge v_8)$. Since a stabilizing matrix of the mentioned form already maps $v_3 \wedge v_4$ to $\mu v_3 \wedge v_4$, we only require that it maps $\widetilde{w_4} := v_5 \wedge v_6 + v_7 \wedge v_8$ to $\lambda^{-1}\widetilde{w_4}$. We can interpret $\widetilde{w_4}$ as an alternating bilinear form in the following way.

Let U be the 4-dimensional vector space over $\mathbb{F}_q$ whose generators $u_1, \ldots, u_4$ correspond to $v_5, \ldots, v_8$ respectively. Now $\Lambda^4 U$ can be identified with the field $\mathbb{F}_q$ in the following natural way:

$$\alpha u_i \wedge u_j \wedge u_k \wedge u_l = \begin{cases} \alpha & i,j,k,l \text{ are all different} \\ 0 & \text{otherwise} \end{cases} \qquad \alpha \in \mathbb{F}_q, \quad i,j,k,l = 1, \ldots, 4$$

Let $\widetilde{w_4}\colon U \times U \to \mathbb{F}_q$ act on $U \times U$ by $\widetilde{w_4}(\alpha u_i, \beta u_j) \mapsto \alpha\beta(u_1 \wedge u_2 + u_3 \wedge u_4) \wedge u_i \wedge u_j$. This map is clearly bilinear due to the multilinear character of the wedge product, and since $u_i \wedge u_j = -u_j \wedge u_i$ it is also alternating.

The stabilizer of a bilinear alternating form is well known and is just (defined as) the symplectic group. (See e.g. [7]). Thus the stabilizer of $\tilde{v}$ is $\mathrm{Sp}_4(q)$. It follows that

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_4\rangle) = \left\{ \begin{pmatrix} \lambda M & *_{2,2} & *_{2,4} \\ 0_{2,2} & \mu \widehat{M} & 0_{2,4} \\ 0_{4,2} & *_{4,2} & \nu M' \end{pmatrix} \middle| \begin{array}{l} M \in \mathrm{SL}_2(q), M' \in \mathrm{Sp}_4(q), \\ \lambda, \mu, \nu \in \mathbb{F}_q^\times, \nu^4 = \lambda^{-1} \end{array} \right\}$$

and hence that

$$|\mathrm{Stab}_{\mathrm{SL}_8(q)}(w_4)| = (q-1)^2 q^{20} |\mathrm{SL}_2(q)| \cdot |\mathrm{Sp}_4(q)| = (q-1)^2 q^{25}(q^2-1)^2(q^4-1).$$

- The element $x_{-\alpha_{53}} + x_{\alpha_{27}} + x_{\alpha_{17}} + x_{-\alpha_4} + x_{\alpha_{50}} + x_{-\alpha_{33}} + x_{-\alpha_{22}} + x_{-\alpha_9}$ corresponds to the element $w_5 = \overline{w_5} + \underline{w_5}$ where $\overline{w_5} := E_{18} + E_{27} + E_{36} + E_{45}$ and $\underline{w_5} := v_1 \wedge v_5 \wedge v_6 \wedge v_7 + v_2 \wedge v_5 \wedge v_6 \wedge v_8 + v_3 \wedge v_5 \wedge v_7 \wedge v_8 + v_4 \wedge v_6 \wedge v_7 \wedge v_8$. From (8) we know that the stabilizer of $\langle \overline{w_5} \rangle$ is given by

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle \overline{w_5}\rangle) = \left\{ \begin{pmatrix} \lambda M & *_{4,4} \\ 0_{4,4} & \lambda^{-1}\widehat{M} \end{pmatrix} \middle| \begin{array}{l} M \in \mathrm{SL}_4(q), \\ \lambda \in \mathbb{F}_q^\times \end{array} \right\},$$

where $\widehat{M}$ is the matrix obtained from the matrix M as defined in Lemma 8.1. Note that matrices of this form map $\overline{w_5}$ to $\lambda^{-2}\overline{w_5}$. Now we look at those matrices within this stabilizer that map $\underline{w_5}$ to $\lambda^{-2}\underline{w_5}$.

Claim 10.1 *The matrices of the form*

$$\left\{ \begin{pmatrix} \lambda M & 0_{4,4} \\ 0_{4,4} & \lambda^{-1}\widehat{M} \end{pmatrix} \middle| M \in \mathrm{SL}_4(q), \lambda \in \mathbb{F}_q^\times \right\} \qquad (17)$$

stabilize $\langle v \rangle$.

Proof: Suppose we have a matrix as in (17) where $M = (m_{ij})_{1\le i,j\le 4}$.
We already know that this matrix maps $\overline{w_5}$ to $\lambda^{-2}\overline{w_5}$ so in order for it to stabilize $\langle w_5\rangle$ it also needs to map $\underline{w_5}$ to $\lambda^{-2}\underline{w_5}$. Hence it needs to satisfy the condition in (14) for each basis element of $\Lambda^4\overline{V}$. Note that all of the determinants of the 4×4 submatrices that occur in these conditions have a factor $\lambda\lambda^{-3} = \lambda^{-2}$, which we divide out to obtain the following simpler conditions.

– Consider the condition for the basis element $v_1\wedge v_5\wedge v_6\wedge v_7$ (which is one of the basis elements occurring in $\underline{w_5}$):

$$\begin{matrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{44} & m_{43} & m_{42} \\ 0 & m_{34} & m_{33} & m_{32} \\ 0 & m_{24} & m_{23} & m_{22} \end{matrix} + \begin{matrix} m_{21} & 0 & 0 & 0 \\ 0 & m_{44} & m_{43} & m_{42} \\ 0 & m_{34} & m_{33} & m_{32} \\ 0 & m_{14} & m_{13} & m_{12} \end{matrix} +$$
$$\begin{matrix} m_{31} & 0 & 0 & 0 \\ 0 & m_{44} & m_{43} & m_{42} \\ 0 & m_{24} & m_{23} & m_{22} \\ 0 & m_{14} & m_{13} & m_{12} \end{matrix} + \begin{matrix} m_{41} & 0 & 0 & 0 \\ 0 & m_{34} & m_{33} & m_{32} \\ 0 & m_{24} & m_{23} & m_{22} \\ 0 & m_{14} & m_{13} & m_{12} \end{matrix} = 1$$

Since our calculations are all over a field of characteristic 2, the left hand side is exactly $\det(M)$. Since $M\in \mathrm{SL}_4(q)$ this condition indeed holds. In a similar way one easily checks that the conditions imposed by looking at the other basis elements occurring in $\underline{w_5}$ (i.e. $v_2\wedge v_5\wedge v_6\wedge v_8, v_3\wedge v_5\wedge v_7\wedge v_8$, and $v_4\wedge v_6\wedge v_7\wedge v_8$) also hold, since in all these conditions, the left hand side can be rewritten as the determinant of the matrix M, which is 1 by our choice of M.

– Consider the condition for the basis element $v_1\wedge v_6\wedge v_7\wedge v_8$ not appearing in $\underline{w_5}$:

$$\begin{matrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{44} & m_{43} & m_{41} \\ 0 & m_{34} & m_{33} & m_{31} \\ 0 & m_{24} & m_{23} & m_{21} \end{matrix} + \begin{matrix} m_{21} & 0 & 0 & 0 \\ 0 & m_{44} & m_{43} & m_{41} \\ 0 & m_{34} & m_{33} & m_{31} \\ 0 & m_{14} & m_{13} & m_{11} \end{matrix} +$$
$$\begin{matrix} m_{31} & 0 & 0 & 0 \\ 0 & m_{44} & m_{43} & m_{41} \\ 0 & m_{24} & m_{23} & m_{21} \\ 0 & m_{14} & m_{13} & m_{11} \end{matrix} + \begin{matrix} m_{41} & 0 & 0 & 0 \\ 0 & m_{34} & m_{33} & m_{31} \\ 0 & m_{24} & m_{23} & m_{21} \\ 0 & m_{14} & m_{13} & m_{11} \end{matrix} = 0$$

This time the left hand side is easily identified with the determinant

$$\begin{matrix} m_{41} & m_{44} & m_{43} & m_{41} \\ m_{31} & m_{34} & m_{33} & m_{31} \\ m_{21} & m_{24} & m_{23} & m_{21} \\ m_{11} & m_{14} & m_{13} & m_{11} \end{matrix}$$

which is clearly 0 since the first and the last column are equal. Thus this condition is satisfied. In a similar way one can check the conditions hold for all basis elements not appearing in $\underline{w_5}$.

□

Thus if we write $S := \mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_5\rangle)$, then

$$\left\{ \begin{pmatrix} \lambda M & 0_{4,4} \\ 0_{4,4} & \lambda^{-1}\widehat{M} \end{pmatrix} \middle| \begin{array}{l} M\in \mathrm{SL}_4(q), \\ \lambda\in \mathbb{F}_q^{\times} \end{array} \right\} \le S \le \left\{ \begin{pmatrix} \lambda M & *_{4,4} \\ 0_{4,4} & \lambda^{-1}\widehat{M} \end{pmatrix} \middle| \begin{array}{l} M\in \mathrm{SL}_4(q), \\ \lambda\in \mathbb{F}_q^{\times} \end{array} \right\}.$$

Now suppose that there exist a matrix $A := (a_{ij})_{1\le i,j\le 4}\neq 0$, a matrix $M\in \mathrm{SL}_4(q)$ and an element $\lambda\in\mathbb{F}_q^{\times}$ such that $\begin{pmatrix} \lambda M & A\widehat{M} \\ 0 & \lambda^{-1}\widehat{M} \end{pmatrix} \in \mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_5\rangle)$. Since $\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_5\rangle)$ is a group, this holds if and only if

$$\left(\begin{array}{cc} \lambda M & A\widehat{M} \\ 0 & \lambda^{-1}\widehat{M} \end{array}\right)\left(\begin{array}{cc} \lambda^{-1}M^{-1} & 0 \\ 0 & \lambda\widehat{M}^{-1} \end{array}\right) = \left(\begin{array}{cc} I & \lambda A \\ 0 & I \end{array}\right) \in \mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_5\rangle).$$

We focus on a matrix of this form and obtain restrictions on A.
In a similar way as in the proof of claim 10.1, one can show that A automatically satisfies all relations imposed by the conditions in (14), except for one. The coefficient of $v_5 \wedge v_6 \wedge v_7 \wedge v_8$ needs to be zero and this is not automatically taken care of. This condition translates to

$$\begin{aligned} 0 &= \left|\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right| + \left|\begin{array}{cccc} a_{21} & a_{22} & a_{23} & a_{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right| + \\ & \left|\begin{array}{cccc} a_{31} & a_{32} & a_{33} & a_{34} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right| + \left|\begin{array}{cccc} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right| \\ &= a_{14} + a_{23} + a_{32} + a_{41}. \end{aligned}$$

Note that the factor λ is of no importance here. Thus the stabilizer of $\langle v\rangle$ is given by

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_5\rangle) = \left\{ \left(\begin{array}{cc} \lambda M & A\widehat{M} \\ 0_{4,4} & \lambda^{-1}\widehat{M} \end{array}\right) \middle| \begin{array}{l} M \in \mathrm{SL}_4(q), A \in M_{4,4}(q), \\ \lambda \in \mathbb{F}_q^\times, A_{14} + A_{23} + A_{32} + A_{41} = 0 \end{array} \right\}$$

and

$$|\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_5\rangle)| = (q-1)q^{15}|\mathrm{SL}_4(q)| = (q-1)q^{21}(q^2-1)(q^3-1)(q^4-1).$$

- The element $x_{\alpha_2} + x_{-\alpha_2}$ corresponds to the element $w_6 = v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_5 \wedge v_6 \wedge v_7 \wedge v_8$.

Claim 10.2 *The stabilizer in* $\mathrm{SL}_8(q)$ *of* $\langle w_6\rangle$ *is given by*

$$Stab_{\mathrm{SL}_8(q)}(\langle w_6\rangle) = \left\{ \left(\begin{array}{cc} M & 0_{4,4} \\ 0_{4,4} & M' \end{array}\right) \middle| \; M, M' \in \mathrm{SL}_4(q) \right\} \cdot < \left(\begin{array}{cc} 0_{4,4} & I_4 \\ I_4 & 0_{4,4} \end{array}\right) > .$$

and thus

$$|Stab_{\mathrm{SL}_8(q)}(\langle w_6\rangle)| = |\mathrm{SL}_4(q)|^2 = 2q^{12}(q^2-1)^2(q^3-1)^2(q^4-1)^2.$$

Proof: Let $M \in SL_8(q)$ such that M maps w_6 to λw_6 for some $\lambda \in \mathbb{F}_q^\times$. Write

$$M = \left(\begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array}\right)$$

and let d_i be the determinant of M_i for $i = 1, 2, 3, 4$. Then from the conditions in (15) we know that $d_1 + d_3 = \lambda$ and $d_2 + d_4 = \lambda$.

We first assume that $d_1 = d_2 = 0$. By developing the determinant $\det(M)$ to rows $5, 6, 7$, and 8 it can be written as a sum of the determinants of 4×4 submatrices of the first four rows of M, but these are all zero, hence so is $\det(M)$. This is a contradiction with the choice of M. In a similar way one can show that it is impossible that $d_3 = d_4 = 0$. Thus at least M_1 and M_4 are both invertible or M_2 and M_3 are both invertible.

Now assume that $d_1 \neq 0$ and $d_4 \neq 0$. We define M' by

$$M' := \left(\begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array}\right)\left(\begin{array}{cc} \delta_1 M_1^{-1} & 0_{4,4} \\ 0_{4,4} & \delta_4 M_4^{-1} \end{array}\right) = \left(\begin{array}{cc} \delta_1 I_4 & A \\ B & \delta_4 I_4 \end{array}\right),$$

where $\delta_1, \delta_4 \in \mathbb{F}_q$ such that $\delta_1^4 = d_1$ and $\delta_4^4 = d_4$ and where $A = \delta_4 M_2 M_4^{-1}$ and $B = \delta_1 M_3 M_1^{-1}$. Since the matrices $\delta_1 M_1^{-1}$ and $\delta_4 M_4^{-1}$ both have determinant 1, one easily sees that M maps w_6 to λw_6 if and only if M' maps w_6 to λw_6. We can express each entry a_{ij} of the matrix A in entries of the matrix B, by using some of the conditions in (15) that we did not exploit yet. Writing out the condition that is imposed by the set of columns $\{1, 2, 3, 4, 4+i\} \setminus \{i\}$, we can immediately relate a_{ij} to a minor of B in the following way:

$$a_{ij} = \delta_1^{-3} \delta_4 (-1)^{i+j} \det(B_{\hat{j}\hat{i}}) \text{ for all } 1 \leq i, j \leq n.$$

Here $B_{\hat{j}\hat{i}}$ denotes the matrix B without the j'th column and the i'th row. Hence A turns out to be a scalar multiple of the *adjunct matrix* $\text{adj}(B)$ of B. (See e.g. [26])

If we write out the condition imposed by the set of columns $\{i, 5, 6, 7, 8\} \setminus (4+i)$, we can relate b_{ij} to a minor of A in a similar way. Thus

$$A = \delta_1^{-3} \delta_4 \text{adj}(B) \text{ and } B = \delta_1 \delta_4^{-3} \text{adj}(A).$$

Now assume that both A and B are non-singular matrices. Then $\text{adj}(A) = \det(A) A^{-1}$ and $\text{adj}(B) = \det(B) B^{-1}$. But then $\det(A) = \det(\delta_1^{-3} \delta_4 \text{adj}(B)) = \delta_1^{-12} \delta_4^4 \det(B)^3$ and in a similar way $\det(B) = \delta_1^4 \delta_4^{-12} \det(A)^3$, hence $\det(A) = \delta_1^{-12} \delta_4^4 (\delta_1^4 \delta_4^{-12})^3 \det(A)^9 = \delta_4^{-32} \det(A)^9$. It follows that $\det(A) = 0$ or $\det(A) = \delta_4^4$. However, $\det(A) = 0$ contradicts our assumption that A is non-singular. Hence $\det(A) = \delta_4^4$ and in a similar way we obtain $\det(B) = \delta_1^4$. But now

$$\det(M') = \begin{vmatrix} \delta_1 I_4 & A \\ B & \delta_4 I_4 \end{vmatrix} = \begin{vmatrix} \delta_1 I_4 & \delta_1^{-3} \delta_4 \det(B) B^{-1} \\ B & \delta_4 I_4 \end{vmatrix} = \begin{vmatrix} \delta_1 I_4 & \delta_1 \delta_4 B^{-1} \\ B & \delta_4 I_4 \end{vmatrix} = 0$$

which contradicts the non-singularity of M'. Thus our assumption that both A and B are non-singular is false and at least A or B has to be a singular matrix. Without loss of generality we assume that $\det(A) = 0$. We now use the following lemma:

Lemma 10.1 *Let $A, B \in M_{n,n}(\mathbb{K})$ for some field $\mathbb{K}$ and some integer $n > 2$ such that $\det(A) = 0$. Let $A = C_1 \text{adj}(B)$ and let $B = C_2 \text{adj}(A))$ for some scalars $C_1, C_2 \in \mathbb{K}^\times$. Then $A = 0$ and $B = 0$.*

Proof: Since $\det(A) = 0$ we have $\text{rk}(A) < n$. If $\text{rk}(A) < n-1$ then A has at least two rows that are linearly dependent of the other $n-2$ rows. But that means that all the $(n-1)$ dimensional sub matrices of A are singular, and hence that $B = C_2 \text{adj}(A) = 0$. Thus clearly $A = C_1 \text{adj}(0) = 0$ and thus also $B = 0$.

If $\text{rk}(A) = n-1$ then B has at least a $n-1$-dimensional nullspace since $\text{adj}(A) A = \det(A) I = 0$. But then $\text{rk}(B) \leq 1 < n-1$. Hence $A = C_1 \text{adj}(B) = 0$ and thus $B = 0$. □ *(lemma 10.1)*

From this lemma, it follows immediately that $A = B = 0$ and hence that $M_2 = M_3 = 0$. Now $\det(M) = \det(M_1) \det(M_3) = \lambda^2$ thus $\lambda = 1$ since $M \in \text{SL}_8(q)$. Thus the matrices of the form

$$\left\{ \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix} \middle| M_1, M_4 \in \text{SL}_4(q) \right\}$$

stabilize v. Now if $\det(M_2) \neq 0$ and $\det(M_3) \neq 0$ then one can show in a similar way that the matrices of the form

$$\left\{ \begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix} \middle| M_2, M_3 \in \text{SL}_4(q) \right\}$$

stabilize w_6. These matrices can be obtained from the previous ones, by right multiplication with the element $\begin{pmatrix} 0_{4,4} & I_4 \\ I_4 & 0_{4,4} \end{pmatrix}$.

Since we covered all possible cases, the stabilizer in $\text{SL}_8(q)$ of w_6 consists of the matrices described above, hence the claim is proven. □

- Consider the element $w_7 = \xi v_1 \wedge v_2 \wedge v_3 \wedge v_4 + \xi^q v_5 \wedge v_6 \wedge v_7 \wedge v_8$ where $\xi \in \mathbb{F}_{q^2}^\times$. Then $w_7 \in \Lambda^4 V'$ where $V' := \mathbb{F}_{q^2}^8$ and V' has the same basis $v_1, \dots, v_8$ as V. Note that for $\xi \in \mathbb{F}_q$ we have $\xi^q = \xi$ and the element w_7 is a scalar multiple of $v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_5 \wedge v_6 \wedge v_7 \wedge v_8$ hence they have the same stabilizer. However, if $\xi \notin \mathbb{F}_q$ then w_7 is different from w_6. We call w_7 a *twisted* element. We are interested in this element w_7 because there exists an isomorphism under which w_7 can be interpreted as an element of $\Lambda^4 V$. We first introduce the necessary automorphisms to be able to perform the twist under which this isomorphism holds.

Let τ be the involution that acts on V' by right multiplication with the matrix N_1, where

$$N_\zeta = \begin{pmatrix} 0_{4,4} & \hat{\zeta}^{q-1} I_4 \\ \hat{\zeta}^{1-q} I_4 & 0_{4,4} \end{pmatrix} \text{ for } \hat{\zeta} \text{ determined by } \hat{\zeta}^4 = \zeta, \qquad \zeta \in \mathbb{F}_{q^2}.$$

Since τ permutes the basis elements $v_1, \dots, v_8$ of V', we can write the action of τ on the indices. Then τ has a natural action on $\Lambda^4 V'$; it takes an element $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4}$ to $v_{i_1 \tau} \wedge v_{i_2 \tau} \wedge v_{i_3 \tau} \wedge v_{i_4 \tau}$. Moreover τ also has a natural action on $\mathrm{SL}_8(V')$ by conjugation with the matrix N_1.

Next, let σ be the Frobenius map defined on $\mathbb{F}_{q^2}$ by

$$\sigma \colon x \mapsto x^q.$$

This map σ acts elementwise on the matrices in the matrix algebra $M_{8,8}(q^2)$. The map σ is clearly an involution since $x^{q^2} = x$ for all $x \in \mathbb{F}_{q^2}$. Since σ and τ commute, $\sigma\tau$ is also an involution. For every mathematical object $\mathcal{S}$ on which $\sigma\tau$ has an action, we denote the fixed points of $\mathcal{S}$ under $\sigma\tau$ by $\mathcal{S}_{\sigma\tau}$.

We now determine the set of $\sigma\tau$-fixed points of $\Lambda^4 V'$. In the following $\mathcal{I}$ denotes the set of all 70 4-tuples $I = \{i_1, i_2, i_3, i_4\}$ from $\{1, 2, 3, 4, 5, 6, 7, 8\}$, such that $i_1 < i_2 < i_3 < i_4$. Let $\mathcal{I}'$ be a subset of $\mathcal{I}$ of size 35, such that $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}'^\tau$. Such a subset clearly exists since τ is an involution that maps each element from $\Lambda^4 V'$ to a unique complementary element.

$$\begin{aligned}
(\Lambda^4 V')_{\sigma\tau} &= (\{ \sum_{I \in \mathcal{I}} \xi_I v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \mid \xi_I \in \mathbb{F}_{q^2} \})_{\sigma\tau} \\
&= \{ \sum_{I \in \mathcal{I}} \xi_I v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} \mid \xi_I \in \mathbb{F}_{q^2}, \xi_I^\sigma = \xi_{I\tau} \text{ for all } I \in \mathcal{I} \} \\
&= \{ \sum_{I \in \mathcal{I}'} \xi_I v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge v_{i_4} + \xi_I^\sigma v_{i_1\tau} \wedge v_{i_2\tau} \wedge v_{i_3\tau} \wedge v_{i_4\tau} \mid \xi_I \in \mathbb{F}_{q^2} \}
\end{aligned}$$

Note that $w_7 \in (\Lambda^4 V')_{\sigma\tau}$. It now follows that $|(\Lambda^4 V')_{\sigma\tau}| = (q^2)^{35} = q^{70}$. Since we also have $|\Lambda^4 V| = q^{70}$, there exists an isomorphism between $(\Lambda^4 V')_{\sigma\tau}$ and $\Lambda^4 V$ under which we can interpret w_7 as an element from $\Lambda^4 V$.

Now let the coefficient ξ in w_7 be arbitrary but fixed in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then the stabilizer in $\mathrm{SL}_8(q^2)$ of $\langle w_7 \rangle$ is given by

$$\mathrm{Stab}_{\mathrm{SL}_8(q^2)}(\langle w_7 \rangle) = \left\{ \begin{pmatrix} M & 0_{4,4} \\ 0_{4,4} & M' \end{pmatrix} \middle| \; M, M' \in \mathrm{SL}_4(q^2) \right\} \cdot < N_\xi > .$$

The proof is completely analogous to that of claim 10.2, except for that we need N_ξ here so that the coefficients ξ and ξ^q are switched around using the Frobenius map whenever τ

interchanges $v_1 \wedge v_2 \wedge v_3 \wedge v_4$ and $v_5 \wedge v_6 \wedge v_7 \wedge v_8$. Clearly N_ξ is invariant under $\sigma\tau$, so

$$\begin{aligned}
&(\mathrm{Stab}_{\mathrm{SL}_8(q^2)}(\langle w_7 \rangle))_{\sigma\tau} \\
= \;& \left\{ \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \middle| M, M' \in \mathrm{SL}_4(q^2)) \right\}_{\sigma\tau} \cdot < N_\xi > \\
= \;& \left\{ \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \middle| M, M' \in \mathrm{SL}_4(q^2)), \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} = \begin{pmatrix} (M')^\sigma & 0 \\ 0 & M^\sigma \end{pmatrix} \right\} \cdot < N_\xi > \\
= \;& \left\{ \begin{pmatrix} M & 0 \\ 0 & M^\sigma \end{pmatrix} \middle| M \in \mathrm{SL}_4(q^2)) \right\} \cdot < N_\xi > \\
\cong \;& \mathrm{SL}_4(q^2) \cdot 2.
\end{aligned}$$

Notice that indeed $\det(M^\sigma) = (\det(M))^q = 1$, since the determinant is a polynomial function in the matrix entries and σ acts as a homomorphism on these entries.

We need the result of the following claim to find the stabilizer in $\mathrm{SL}_8(q)$ of $\langle w_7 \rangle$, where $\mathrm{SL}_8(q)$ acts through the isomorphism described in (18).

Claim 10.3

$$\mathit{Stab}_{\mathrm{SL}_8(q)}(v) \cong (\mathit{Stab}_{\mathrm{SL}_8(q^2)}(v))_{\sigma\tau}.$$

Proof: First note that $(\mathrm{Stab}_{\mathrm{SL}_8(q^2)}(v))_{\sigma\tau} = \mathrm{Stab}_{\mathrm{SL}_8(q^2)_{\sigma\tau}}(v)$. Thus we only need to prove that $\mathrm{SL}_8(q^2)_{\sigma\tau} \cong \mathrm{SL}_8(q)$.

We can characterize $\mathrm{SL}_8(q)$ and $\mathrm{SL}_8(q^2)$ by

$$\mathrm{SL}_8(q) = \{ \, g \in G \mid g^\sigma = g \, \}$$

and

$$\mathrm{SL}_8(q^2)_{\sigma\tau} = \{ \, g \in G \mid g^{\sigma\tau} = g \, \}$$

Lang's theorem now states that for each $x \in \mathrm{SL}_8(q^2)$, the map $x \mapsto x^{-1}x^\sigma$ is surjective. (See e.g. [30],[14].) But that means that there exists an $x \in G$ such that $x^{-1}x^\sigma = \tau^{-1}$. Consider this x. Then $x^{-1}x^\sigma\tau = 1$, thus for all $a \in \mathrm{SL}_8(q^2)$ we have that $x^{-1}x^\sigma\tau a = ax^{-1}x^\sigma\tau$, hence that $a^{\tau^{-1}\sigma^{-1}x^{-1}\sigma} = a^{x^{-1}}$. Using this equality, we see that

$$a^{\sigma\tau} = a \Leftrightarrow a^\sigma = a^{\tau^{-1}} \Leftrightarrow a^{x^{-1}\sigma} = a^{\tau^{-1}\sigma^{-1}x^{-1}\sigma} = a^{x^{-1}}.$$

Thus if a is stabilized under $\sigma\tau$ then $a^{x^{-1}}$ is stabilized under σ, hence

$$\mathrm{SL}_8(q^2)_{\sigma\tau} = (\mathrm{SL}_8(q))^x \tag{18}$$

which shows that these two groups are conjugate under an action from $\mathrm{SL}_8(q^2)$ hence they are definitely isomorphic. □

Thus the stabilizer of the element w_7, which can be interpreted as an element of $\Lambda^4 V$, is given by

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_7 \rangle) \cong \mathrm{SL}_4(q^2) \cdot 2$$

and

$$|\mathrm{Stab}_{\mathrm{SL}_8(q)}(\langle w_7 \rangle)| = 2q^{12}(q^4-1)(q^6-1)(q^8-1).$$

Determining the $\tilde{H}$-stabilizers

Note that we are not directly interested in the stabilizers of the elements $w_1, \ldots, w_7$ under the action of $\mathrm{SL}_8(q)$, but that we wish to find the stabilizers under the action of $\tilde{H}$. The action of ς on the elements $w_1, \ldots, w_7$ is shown in table 4. From this table and from the way the explicit $\mathrm{SL}_8(q)$-stabilizers are computed, it is immediately clear that

$$\mathrm{Stab}_{\mathrm{SL}_8(q)}(w_i^\varsigma) \cong \mathrm{Stab}_{\mathrm{SL}_8(q)}(w_i) \text{ for all } i = 1, \ldots, 7.$$

Hence it follows that

$$\mathrm{Stab}_{\tilde{H}}(w_i) \cong \mathrm{Stab}_{\mathrm{SL}_8(q)}(w_i) \cdot 2 \text{ for all } i = 1, \ldots, 7.$$

i	w_i	w_i^{ς}
1	$E_{1,2}$	$E_{2,1}$
2	$v_1 \wedge v_2 \wedge v_3 \wedge v_4$	$v_5 \wedge v_6 \wedge v_7 \wedge v_8$
3	$E_{1,2} + v_2 \wedge v_3 \wedge v_4 \wedge v_5$	$E_{2,1} + v_1 \wedge v_6 \wedge v_7 \wedge v_8$
4	$E_{1,4} + E_{2,3}+$ $v_3 \wedge v_4 \wedge v_5 \wedge v_6 + v_3 \wedge v_4 \wedge v_7 \wedge v_8$	$E_{4,1} + E_{3,2}+$ $v_1 \wedge v_2 \wedge v_7 \wedge v_8 + v_1 \wedge v_2 \wedge v_5 \wedge v$
5	$v_1 \wedge v_5 \wedge v_6 \wedge v_7 + v_2 \wedge v_5 \wedge v_6 \wedge v_8+$ $v_3 \wedge v_5 \wedge v_7 \wedge v_8 + v_4 \wedge v_6 \wedge v_7 \wedge v_8$	$v_2 \wedge v_3 \wedge v_4 \wedge v_8 + v_1 \wedge v_3 \wedge v_4 \wedge v_7+$ $v_1 \wedge v_2 \wedge v_4 \wedge v_6 + v_1 \wedge v_2 \wedge v_3 \wedge v_5$
6	$v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_5 \wedge v_6 \wedge v_7 \wedge v_8$	$v_5 \wedge v_6 \wedge v_7 \wedge v_8 + v_1 \wedge v_2 \wedge v_3 \wedge v_4$
7	$\xi v_1 \wedge v_2 \wedge v_3 \wedge v_4 + \xi^q v_5 \wedge v_6 \wedge v_7 \wedge v_8$	$\xi v_5 \wedge v_6 \wedge v_7 \wedge v_8 + \xi^q v_1 \wedge v_2 \wedge v_3 \wedge v_4$

Table 4: The action of ς on the elements $w_1, \ldots, w_7$.

11 Summary of the results

In section 10 we determined the $\tilde{H}$ stabilizers of 7 distinct elements $w_1, \ldots, w_7$ which can be interpreted as elements from the set Y through the isomorphism γ (see section 7.3 and table 10). A description of the elements and their stabilizers can be found in table 5.

Name	Description of w_i	Description of $\mathrm{Stab}_{\tilde{H}}(\langle w_i \rangle)$
w_1	$E_{1,2}$	$\mathbb{F}_q^{13}(\mathbb{F}_q^\times)^2\mathrm{SL}_6(q)\mathbb{Z}_2$
w_2	$v_1 \wedge v_2 \wedge v_3 \wedge v_4$	$\mathbb{F}_q^{16}\mathbb{F}_q^\times\mathrm{SL}_4(q)\mathrm{SL}_4(q)\mathbb{Z}_2$
w_3	$E_{1,2} + v_2 \wedge v_3 \wedge v_4 \wedge v_5$	$\mathbb{F}_q^{22}(\mathbb{F}_q^\times)^2\mathrm{SL}_3(q)\mathrm{SL}_3(q)\mathbb{Z}_2$
w_4	$E_{1,4} + E_{2,3} + v_3 \wedge v_4 \wedge v_5 \wedge v_6 + v_3 \wedge v_4 \wedge v_7 \wedge v_8$	$\mathbb{F}_q^{20}(\mathbb{F}_q^\times)^2\mathrm{SL}_2(q)\mathrm{Sp}_4(q)\mathbb{Z}_2$
w_5	$v_1 \wedge v_5 \wedge v_6 \wedge v_7 + v_2 \wedge v_5 \wedge v_6 \wedge v_8+$ $v_3 \wedge v_5 \wedge v_7 \wedge v_8 + v_4 \wedge v_6 \wedge v_7 \wedge v_8$	$\mathbb{F}_q^{15}\mathbb{F}_q^\times\mathrm{SL}_4(q)\mathbb{Z}_2$
w_6	$v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_5 \wedge v_6 \wedge v_7 \wedge v_8$	$(\mathrm{SL}_4(q)\mathrm{SL}_4(q))\mathbb{Z}_2\mathbb{Z}_2$
w_7	$\xi v_1 \wedge v_2 \wedge v_3 \wedge v_4 + \xi^q v_5 \wedge v_6 \wedge v_7 \wedge v_8$, where $\xi \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$	$(\mathrm{SL}_4(q^2))\mathbb{Z}_2\mathbb{Z}_2$

Table 5: The different stabilizers of elements from Y under the action of H.

These stabilizers correspond to 7 H-stabilizers of distinct elements from Y. Once we know these stabilizers and their orders, it is straightforward to compute the sizes of the 7 H orbits, corresponding to these stabilizers. (Using the Orbit-Stabilizer lemma.) The results are shown in table 6, where we interpret the elements w_i as elements from Y through the isomorphism γ. We write $P_n = 1 + q + q^2 + \ldots + q^n$ as an abbreviation.

In section 5 the set Y is identified with the set of right cosets of K in G. Since $G = E_7(q)$ and K is the stabilizer of $\mathbb{F}_q x_{\alpha_0}$ in G, the group K is easily seen to be the parabolic subgroup of type D_6 in $E_7(q)$. Thus $K = D_6(q)_{\mathrm{parab}}$. The orders of both K and H are well known and given by

$$|E_7(q)| = \frac{1}{(2, q-1)} q^{63}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$$

and

$$|D_6(q)_{\mathrm{parab}}| = \frac{1}{(2, q-1)^2} q^{63}(q-1)(q^2-1)(q^4-1)(q^6-1)(q^8-1)(q^{10}-1)(q^6-1).$$

Name	$\lvert H/\mathrm{Stab}_H(\langle w_i\rangle)\rvert$	Degree
w_1	$(q+1)(q^2+1)(q^4+1)P_6$	13
w_2	$(q^2-q+1)(q^4+1)P_4P_6$	16
w_3	$(q-1)(q+1)^2(q^2-q+1)(q^2+1)^2(q^4+1)P_4P_6$	23
w_4	$(q-1)^2q^3(q+1)(q^2-q+1)(q^2+1)(q^2+q+1)^2(q^4+1)P_4P_6$	28
w_5	$(q-1)^3q^7(q+1)^2(q^2-q+1)(q^2+1)(q^2+q+1)(q^4+1)P_4P_6$	32
w_6	$(q-1)q^{16}(q^2-q+1)(q^4+1)P_4P_6$	33
w_7	$(q-1)^4q^{16}(q+1)(q^2+q+1)P_4P_6$	33

Table 6: The sizes of the 7 H orbits on Y

Since $q = 2^p$ for some p, the initial factors return 1 hence the order of Y is given by

$$|Y| = \frac{(q^{12}-1)(q^{14}-1)(q^{18}-1)}{(q-1)(q^4-1)(q^6-1)}$$

Summing up the orbit sizes $|(w_i)^H|$ for $i = 1, \ldots, 7$ we see that $|Y| = \sum_{i=1}^{7} |(w_i)^H|$. Hence the 7 H-orbits that we found on Y, are the only 7 that exist. This finishes the proof of Theorem 5.3, and by doing so, it shows that the permutation action of $E_7(q)$ on the set of right cosets of $A_7(q)\cdot 2$ in $E_7(q)$) is not multiplicity free. According to Theorem 5.1 this implies that there does not exist a distance-transitive graph with automorphism group $E_7(q)$ and vertex stabilizer $A_7(q) \cdot 2$ (see Corollary 5.1.

11.1 Recommendations for future research

It is conceivable that the results obtained in this thesis, can be applied to investigate another open case in Table 1, namely that of the maximal subgroup $^2\,A_7(q)$ of $E_7(q)$. Note that $^2A_7(q)$ is obtained from $A_7(q^2)$, by looking at the fixed points under the product of a graph automorphism of A_7 and the Frobenius map. There was no time to investigate this in this thesis. However, I did create an explicit construction in Magma of $^2A_7(q)$ as a subgroup of a group isomorphic to $E_7(q)$. This implementation might be useful and can be obtained from the author[9]. Built-in functions in Magma for the construction of twisted groups can be expected in the not too distant future[10].

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[10] by A.M. Cohen, S. E. Haller, S.H. Murray, and D. E. Taylor

Glossary

The first table is a glossary of the used symbols. Note that some symbols may have more than one meaning. From context it will always be clear which interpretation should be used. The second table is a glossary of used notations.

Symbol	Explanation	p.
A_7, D_6, E_7	Lie types	5
v	the element $v = C_{L_{E_7}}(L_{A_7})$	17
G	Lie group of type E_7 over $\mathrm{GF}(q)$	14
H	Lie group of type A_7 over $\mathrm{GF}(q)$, extended by graph automorphism σ_δ	14
L_{E_7}	adjoint Lie algebra of type E_7 over $\mathrm{GF}(q)$ with root system Φ	14
L_{A_7}	adjoint Lie algebra of type A_7 over $\mathrm{GF}(q)$ with root system Ψ	14
N	the anti-diagonal identitymatrix, in particular $N = N_1$	23
N_ζ	matrix holding an element invariant	32
V	natural $\mathrm{SL}_8(q)$ module	20
X	$\{ \mathbb{F}_q v^g \mid g \in H\backslash G \}$	17
Y	$\{ \mathbb{F}_q (x_{\alpha_0})^g \mid g \in G \}$	17
π	permutation character of the G action on X	17
ρ	permutation character of the G action on Y	18
σ	the Frobenius map $x \mapsto x^q$	32
τ	an involution acting on V by rightmultiplication with N_1	32
Π	a fundamental system	4
Φ	root system of type E_7	13
Ψ	root system of type A_7, obtained as a closed set of roots from Φ	14
w_0	the longest Weyl group element	6
ς	$\tilde{\phi}(w_0^\Phi)$, where w_0^Φ is the longest element in $W(\Phi)$	21
γ	the isomorphism as H-modules from $L_{A_7} \oplus W$ to $\mathrm{sl}_8(q) \oplus \Lambda^4 V$	22

Table 7: Explanation of the used symbols. With first time of appearance.

Notation	Explanation	p.
$*$	an arbitrary value from GF(q)	
$*_{r,s}$	an arbitrary $r \times s$ matrix over GF(q)	
$0_{r,s}$	a $r \times s$ all zero matrix	
1_r	a $r \times r$ identity matrix	
E_{ij}	elementary matrix that has only zero entries,	20
	except on the position (i, j), which value is a 1	
$\widehat{A}$	the square matrix, defined by $\widehat{A} = NAN^{-1}$	23
$\mathbb{K}$	an arbitrary field	6
$\mathbb{F}_q, \mathrm{GF}(q)$	a finite field with $q = 2^p$ for some p	14
$\mathbb{F}_q^\times$	$\mathbb{F}_q \setminus 0$	
$SL_d(q)$	the d dimensional special linear group, over $GF(q)$	
$Sp_d(q)$	the d dimensional symplectic group, over $GF(q)$ (d even)	
$GL_d(q)$	the d dimensional general linear group, over $GF(q)$	
$\Lambda^4 V$	the fourth exterior power of V	21
α,α_i,β,β_i	roots	4
x_α	root elements of some Lie algebra	7
$X_\alpha(t)$	standard generators of a Lie group	7
Φ^+, Φ^-	set of positive respectively negative roots of Φ	5
$\langle \alpha, \beta \rangle$	the standard inner product of two roots	
$\langle \chi_1, \chi_2 \rangle$	the inner product of two characters	8
1_G	character belonging to trivial linear representation of G	8
δ_{ij}	the Kronecker delta, returning 1 if $i = j$ and 0 otherwise	20
$\langle v \rangle$	the one dimensional subspace $\mathbb{F}_q v$	
S_σ	the fixed points of a set S under a map σ	32
$G_v, \mathrm{Stab}_G(v)$	the stabilizer of the element v under the action of the group G	9,17
v^G	the G orbit of the element v	3
$H \backslash G$	the set of right cosets of H in G: $\{ Hg \mid g \in G \}$	
$S \setminus T$	the set S without the elements in T: $\{ s \in S \mid s \notin T \}$	4
B	a Borel subgroup of G	
N	a subgroup of G, related to the Weyl group W	8
$W, W(\Phi)$	the Weyl group; the group of reflections of some root space Φ	6
$\Gamma, \Gamma(V, E)$	a graph $\Gamma = \Gamma(V, E)$ with vertex set V and edge set E	3
$\mathrm{d}(v, w)$	the distance between two vertices v and w	3
$D_i(\Gamma)$	$\{ (v, w) \in V \times V \mid \mathrm{d}(v, w) = i \}$	3
$D_i(v)$	$\{ w \in V \mid \mathrm{d}(v, w) = i \}$	3

Table 8: Explanation of used notations. Reference to further explanation is included when relevant.

A Subalgebras and subgroups originating from closed sets of roots

Saying that Ψ is a closed set of roots means that it is a subset of Φ satisfying that for all $\alpha, \beta \in \Psi$ and for all positive natural numbers i, j we have $i\alpha + j\beta \in \Phi$ implies $i\alpha + j\beta \in \Psi$.

Now given a Lie algebra $\mathfrak{L} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is the Cartan subalgebra, restriction of Φ to Ψ gives a subalgebra $\mathfrak{M} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ of $\mathfrak{L}$. It can easily be checked that this is indeed a subalgebra by looking at the generators:

- for all $h_\alpha, h_\beta \in \mathfrak{M}$ we have $[h_\alpha, h_\beta] = 0 \in \mathfrak{M}$ since both are elements of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{M}$;
- for $h_\alpha, x_\beta \in \mathfrak{M}$ we have $[h_\alpha, x_\beta] = A_{\alpha,\beta} x_\beta \in \mathfrak{M}$;
- for $x_\alpha, x_{-\alpha} \in \mathfrak{M}$ we have $[x_\alpha, x_{-\alpha}] = h_\alpha \in \mathfrak{M}$;
- for $x_\alpha, x_\beta \in \mathfrak{M}$ we have $[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Phi \\ N_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \end{cases}$.
  Thus in the first case $0 \in M$ holds and in the second case we have $\alpha + \beta \in \Phi$ and thus $\alpha + \beta \in \Psi$. Therefore $x_{\alpha+\beta} \in \mathfrak{M}$ and thus $[x_\alpha, x_\beta] \in \mathfrak{M}$.

Thus $\mathfrak{M}$ is indeed a subalgebra of $\mathfrak{L}$. This sub-structure is inherited in a natural way by moving to Lie algebras $\mathfrak{L}_\mathbb{K}, \mathfrak{M}_\mathbb{K}$ over an arbitrary field $\mathbb{K}$. It also passes on to the corresponding Chevalley groups. Let $\mathfrak{L}(\mathbb{K})$ be the Chevalley group of type $\mathfrak{L}$ over some field $\mathbb{K}$. Then $\mathfrak{L}(\mathbb{K})$ is generated by the elements $X_\alpha(t) := \exp(t \,\mathrm{ad}\, x_\alpha)$ for all $\alpha \in \Phi$ and all $t \in \mathbb{K}$. Now define $\mathfrak{M}(\mathbb{K})$ to be the group generated by the elements $X_\alpha(t)$ for all $\alpha \in \Psi$ and all $t \in \mathbb{K}$. Then $\mathfrak{M}(\mathbb{K})$ is a subgroup of $\mathfrak{L}(\mathbb{K})$.

B Other approaches

In showing that there does not exist a distance-transitive graph with automorphism group $E_7(q)$ and the stabilizer of a vertex $A_7(q) \cdot 2$ for any $q = 0 \bmod 2$, using character theory is not the only path one can follow, and it is definitely not the first approach we have tried. In this section some other approaches are explained. These approaches were not successful in this particular case, but they might be successful in other cases. They are mostly useful to show non-existence of some distance-transitive graph.

In the remainder of this section, we let G be the automorphism group of some distance-transitive graph $\Gamma = (V, E)$ if it exists. Furthermore, H is the stabilizer of some element $v \in V$.

B.1 The intersection array

With each (finite) graph, we can associate a collapsed adjacency diagram. These diagrams represent the structure of adjacency within a graph. Let $\Gamma = \Gamma(V, E)$ be a graph with automorphism group G. Fix a vertex v of $\mathcal{G}$ and define $H = G_v$. (This is a similar situation as how we set up graphs in section 3.1). The group H (trivially) acts distance-transitive on the H-suborbits of V. Each H-suborbit consists of elements that are at a fixed distance of v. The collapsed adjacency diagram for $\mathcal{G}$ consists of vertices, which we call circles, and directed edges, which we call arrows.

- The circles correspond to the H-suborbits of V, thus to subsets of V. Each circle is labeled by its size.
- The special circle of size 1, corresponding to the set $\{v\}$ is often represented by a small circle or node.
- There is an arrow from circle i to circle j if some (and thus all) vertex in circle i is adjacent to at least 1 vertex in circle j. The arrow is labeled by the number of vertices in circle j, that i is adjacent to.
- There are numbers above the circles that can be interpreted as the labels of an arrow from that circle to itself.

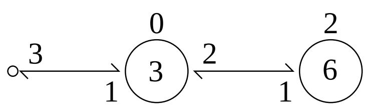
Note that the higher the symmetry of the graph, the simpler the structure of the collapsed adjacency diagram. Looking at two extremes, on one side we find graphs Γ with no symmetry at all, thus for which the automorphism group G is the identity group. Obviously the stabilizer $H := G_v$ of an arbitrary vertex v, is again the identity group and thus the H-suborbits correspond to the vertices of Γ. In this case the collapsed adjacency graph is isomorphic to the original graph.

On the other side we find graphs with a high level of symmetry, the Distance-Transitive Graphs, the object of our interest. For these graphs the collapsed adjacency diagram has a very simple form.

Lemma B.1 *For each Distance-Transitive Graph, its collapsed adjacency diagram is a path.*

Proof: Let (v, u_1) and (v, u_2) be two pairs of vertices of a distance-transitive graph Γ such that $d(v, u_1) = d(v, u_2)$. Then there must exist an element $g \in G$ such that g stabilizes v and $u_1^g = u_2$. But then u_1 and u_2 are in the same G_v orbit thus they are in the same circle in the collapsed adjacency diagram, hence there only exists one circle at each distance from v and the collapsed adjacency diagram is a path. □

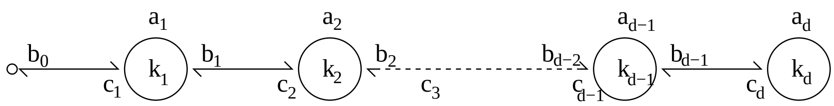
As an example of the collapsed adjacency diagram of a Distance Transitive Graph we give the collapsed adjacency diagram of the Petersen graph.



Note that Distance-Transitive Graphs are not the only graphs for which the collapsed adjacency diagram is a path. This also holds for the more general class of Distance Regular Graphs, of which the Distance-Transitive Graphs are a subset.

All the information in this diagram can be stored as a sequence of numbers, namely of the parameters in the diagram. It is however enough to only store the labels at the (real) arrows. Note that the sizes of the circles and the numbers above them (the labels of the loops) can be easily obtained from these and are thus redundant. We will make this more precise.

Lemma B.2 *Let the collapsed adjacency diagram of a Distance Regular Graph be given by*



where d is the diameter of the (original) graph. This diagram can be uniquely represented by a so-called intersection array

$$(b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d).$$

Proof: Let k denote the valency of the (original) Distance Transitive Graph. We then obtain:

- $a_i = k - b_i - c_i$ for $i = 0, \dots, d$;
- $k_i b_i = k_{i+1} c_{i+1}$ for $i = 0, \dots, d-1$;
- $b_0 = k$ and $c_1 = 1$.

Hence all the a_i's and k_i's can be computed from the information in the intersection array. □

The intersection array of the Petersen graph is $(3, 2; 1, 1)$.

We now give some necessary (but not sufficient) conditions on the parameters in the intersection array, for it to give rise to a Distance Regular Graph.

Let Γ be a Distance Regular Graph of diameter d and valency k with intersection array $I = (b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d)$. Define $k_i := |D_i(v)|$ for an arbitrary but fixed vertex v and for $i = 0, \dots d$. Then I must satisfy the following conditions ([5]):

1. all parameters are non negative integers
2. $k = b_0 > b_1 \geq b_2 \geq \dots \geq b_{d-1} > b_d = 0$
3. $1 = c_1 \leq c_1 \leq c_2 \leq \dots \leq c_d \leq k$
4. if $i + j \leq d$ then $b_i \geq c_j$
5. the sequence $k_0, k_1, \dots, k_d$ is unimodal. This means there exist i, j such that $1 = k_0 < k_1 < \dots < k_i = \dots = k_j > \dots > k_d$. Hence, the largest H-orbit is the only H-orbit that can occur more than twice.

B.2 Invariants

A general technique that is used all throughout group theory, is that of finding suitable invariants that reveal information on the structure of the module of our interest. In the search for distance-transitive graphs, this approach can be used to obtain information on the H-suborbit structure of some vertex set V. The idea is to find functions on the vertex set V that are invariant under the action of H but not invariant under the action of G. In this way we can partition the vertex set in such a way that each suborbit lies entirely within a single partition. By simultaneously looking at more invariants it is possible to make the partitions smaller and therefore find more restrictions on the positions of the suborbits. Since the actual invariants vary a lot from case to case, in this section we discuss some invariants for the particular case we studied. Hence in this section the vertex set is denoted by X, analogously to the notation in the main part of this thesis.

The invariants we look for are all either functions on the set X.

- **Commutation**
  Given an element $w \in X$, v and w commute if and only if $w \in \langle L_{A_7}, v\rangle$. If v and w commute then v and wh commute for all $h \in H$ since $[v,w]=0 \Leftrightarrow [v,wh]=[vh,wh]=[v,w]h=0h=0$. This leads to the invariant

  $$Comm(x) := \text{true iff } [v,x]=0,$$

  defined for $x \in X$.

- **Subalgebra**
  Given an element $w \in X$ we can construct a subalgebra of L_{E_7} generated by the elements v and w. Let's write $S(v,w)$ for this subalgebra. Then $S(vg,wg)=(S(v,w))g$, which is a direct consequence from the fact that right multiplication with an element $g \in G$ respects the Lie bracket. Thus this leads to the invariant

  $$DS(x) := \text{Dimension}(S(v,x)),$$

  defined for $x \in X$. Other invariants are for example the direct sum decomposition of $S(v,w)$ or its nilradical or solvable radical or its centralizer.

- **Centralizer**
  The centralizer $C_{L_{A_7}}(w)$ of w in L_{A_7} is a source for invariants under H. For an arbitrary element $w \in X$ we have by definition $C_{L_{A_7}}(w)=\{\, l \in L_{A_7} \mid [l,w]=0 \,\}$. Fix this w and now consider $C_{L_{A_7}}(wg)$ where $g \in G$. Then

  $$\begin{aligned} C_{L_{A_7}}(wg) &= \{\, l \in L_{A_7} \mid [l,wg]=0 \,\} \\ &= \{\, l \in L_{A_7} \mid [lg^{-1},w]g=0 \,\} \\ &= \{\, l \in L_{A_7} \mid [lg^{-1},w]=0 \,\} \\ &= \{\, kg \in L_{A_7} \mid [k,w]=0 \,\} \\ &= C_{L_{A_7}g^{-1}}(w)g. \end{aligned}$$

  Now for $g \in H$ this simplifies further to $C_{L_{A_7}}(w)g$. Thus a possible invariant is

  $$DC(x) := \text{Dimension}(C_{L_{A_7}}(x)),$$

  defined for $x \in X$. Just as in the previous item regarding the subalgebras, other invariants are for example the direct sum decomposition, the nilradical and the solvable radical.

  Another source for invariants is the centralizer $C_{L_{E_7}}(S(v,w))$. One easily checks that $C_{L_{A_7}}(w) \subseteq C_{L_{E_7}}(S(v,w))$. Remember that $\langle v, L_{A_7}\rangle = C_{L_{E_7}}(v)$ thus this inclusion is trivial. Now equality occurs if and only if there exists no $l \in L_{A_7}$ such that $[v+l,w]=0$. If there is such a $l \in L_{A_7}$ however, then the dimension of $C_{L_{E_7}}(S(v,w))$ will be exactly 1 higher than the dimension of $C_{L_{A_7}}(w)$ since if there would be an $l' \in L_{A_7}, l' \neq l$ such that $[v+l',w]=0$ then also $[v+l'+v+l,w]=[l+l',w]=0$ which implies $l+l' \in L_{A_7}$ and thus $v+l'$ would be linearly dependent of $\langle C_{L_{A_7}}(w), v+l\rangle$. We obtain the invariant

  $$DCS(x) := \text{Dimension}(C_{L_{E_7}}(S(v,x))),$$

  defined for $x \in X$.

- **Conjugacy class of the adjoint matrix**
Remember that for all elements $v, w \in L_{E_7}$ and any automorphism $\rho \in \mathrm{Aut}(L_{E_7})$ we have that $[v\rho, w\rho] = [v, w]\rho$. In particular the elements of the Lie group G act as automorphisms on the Lie algebra L_{E_7}. We also say that right multiplication with an element of G *respects* the Lie bracket. Now let w be an arbitrary element from L_{E_7}. Let moreover $g \in G$. Then $(\mathrm{ad}_{wg})(x) = [wg, x] = [w, xg^{-1}]g = g^{-1}(\mathrm{ad}_w)g(x)$ holds for all $x \in L_{E_7}$. Thus in particular the set X is partitioned by computing conjugacy classes of the adjoint matrices of its elements. We don't use this partitioning directly, since testing for conjugacy of the elements is hard to do. We will use this theory as the basis of a few more invariants.

- **The Characteristic polynomial**
For $x \in X$ we define its characteristic polynomial by

$$CP(x) := \mathrm{Det}(\mathrm{ad}_x - \lambda I).$$

This function returns a polynomial P in λ such that $P(\mathrm{ad}_x) = 0$. It is immediate that $CP(xg) = \mathrm{Det}(\mathrm{ad}_{xg} - \lambda I) = \mathrm{Det}(g^{-1}\mathrm{ad}_x g - \lambda g^{-1} I g) = \mathrm{Det}(g^{-1}(\mathrm{ad}_x - \lambda I)g) = \mathrm{Det}(\mathrm{ad}_x - \lambda I) = CP(x)$ for all $x \in X$ and all $g \in G$. Using the characteristic polynomial in this way will however only create a trivial partitioning on X, namely all of X, containing all vertices in X and thus also all suborbits. In stead of looking at the value of CP in an arbitrary vertex $x \in X$ we can also look at its value in for example the vertices $v + w, [v, w]$ and $v + [v, w]$ where $v = C_{L_{E_7}}(L_{A_7})$ and $w \in X$. Since elements $h \in H$ leave v invariant and trivially move w around in the suborbit under H this *will* partition the set X into non trivial parts. The concrete invariants are then

$$CP_1(x) := CP(v + x), CP_2(x) := CP([v, x]) \text{ and } CP_3(x) := CP(v + [v, x]),$$

defined for $x \in X$.

- **The Jordan Decomposition**
From Humphreys ([18]) we obtain that for all $a \in \mathrm{End}(X)$ there exist unique $a_s, a_n \in \mathrm{End}(X)$ such that $a = a_s + a_n$, a_s is semisimple, a_n is nilpotent, a_s and a_n commute. Moreover there exist polynomials $p(\lambda), q(\lambda)$ in one indeterminate, without a constant term, such that $a_s = p(a), a_n = q(a)$. In particular, a_s and a_n commute with any endomorphism commuting with a. This decomposition $a = a_s + a_n$ is called the additive Jordan-Chevalley decomposition or simply the Jordan decomposition. Since conjugation with an element $g \in G$ of an element in $\mathrm{End}(X)$ leaves nilpotency and semisimplicity invariant we have that the Jordan decomposition of $g^{-1}ag$ will be given by $g^{-1}ag = g^{-1}a_s g + g^{-1}a_n g$ for all $g \in G$. Since $\mathrm{ad}_x \in \mathrm{End}(X)$ for $x \in X$ this now brings us to define

$JD(x) :=$ The polynomial p as defined above with respect to the Jordan decomposition of ad_x

for $x \in X$. It is immediate now that $JD(xg) = JD(x)$ for all $x \in X$ and $g \in G$. Just as with the characteristic polynomial, looking at $JD(x)$ for all $x \in X$ will only give the trivial partitioning of X. Again we introduce three concrete invariants

$$JD_1(x) := JD(v + x), JD_2(x) := JD([v, x]) \text{ and } JD_3(x) := JD(v + [v, x]),$$

defined for $x \in X$.

- **Nilpotency**
Let $a \in \mathrm{End}(X)$ and let its Jordan decomposition be given by $a = a_s + a_n$. Let moreover $a_s = p(a)$ and $a_n = q(a)$ according to the earlier description. Then by definition a is nilpotent if and only if $a_n = 0$, thus if and only if $q(\lambda) = 0$ and thus $p(\lambda) = \lambda$. Although nilpotency is an invariant, it follows from the previous remark that $x \in X$ is nilpotent if and only if $JD(x) = \lambda$, thus the invariance of nilpotency is covered by computing the Jordan decomposition using the functions JD_i defined earlier and need not be considered as a separate invariant.

B.3 Self paired orbits

Finally we discuss a constraint on the H-orbits on the set $V \times V$. The group G acts in a natural way on the set $V \times V$. The orbits on this set of pairs are called *orbitals*. An orbital O is *self-paired* if $(x, y) \in O \Rightarrow (y, x) \in O$. Since we are looking for a distance-transitive graph, which is by definition an undirected graph, we require the edge set E to be self-paired. Note that E is determined uniquely by pointing out an H-suborbit S_1 of V as the neighbour set of an arbitrary but fixed vertex v. Then $(v, w) \in V \times V$ is an edge for all $w \in S_1$. Since the group G acts transitively on the set of edges this determines all of the adjacency relations in the graph. The edge set E can be written as $E = \{ (x, y) \in V \times V \,|\, (x, y) = (v, w)^g \text{ for some } g \in G, w \in S_1 \}$. Now what does it mean to require that the edge set is self-paired? This is equivalent to the condition that (w, v) is an edge for all $w \in S_1$. Thus that there exists a $g \in G$ such that $(v, w)^g = (w, v)$. Let's pick a $w \in S_1$ and assume that we have a $g \in G$ such that $v^g = w$. Since v is stabilized by H this means that $v^{hg} = w$ for all $h \in H$. In order for self-pairedness to hold, we now require $w^{hg} = v$ for some $h \in H$. Or equivalently we require that $v^{g^{-1}} \in S_1$.

It is not enough to check edge set for self-pairedness. The set $D_i(\Gamma) = \{(x, y) \in V \times V | \mathrm{d}(x, y) = i\}$, consisting of all pairs of vertices at distance i of each other, has to be self paired for all $i = 1, \dots, d$, where d is the diameter of the graph. Again D_i is determined uniquely by pointing out a set S_i at distance i of the vector v. G acts transitively on all sets S_i for $i = 1, \dots, d$. Note that each H-orbit is fully contained within in some S_i, but need not be equal to it. Some H-suborbits may be fused in the same set S_i. As a result we cannot be sure yet that the H-suborbits will correspond to different distance sets. Testing for all the H-suborbits whether they lead to a self-paired distance relation with v tells us more.

In case the orbits are small, this can easily be verified by computing the orbit of one of the two elements and apply membership testing to the other element and this orbit. If the orbits are big however, it might be impossible to compute the orbits explicitly. We mention a slightly more efficient algorithm in case we expect two elements to be in the same orbit. In stead of computing the orbit of one of the two elements and checking for membership of the other element in this orbit, we now simultaneously start computing the orbits of both the elements, while we look for an intersection of these orbits. If an intersection is encountered, the two elements were in the same orbit. The overlap will in general be found faster than a full orbit would have been computed. Note however that if the elements are not in the same orbit, this algorithm works slower than computing the full orbit of one of the two elements and checking for membership of the other element in this orbit.

Note that in the case that we study in this thesis, the H-suborbits are too big to be fully computed, so the proposed algorithm seems to be our only (computational) hope for determining if the H-suborbits that we have found explicitly are self-paired. However, since our case is also much too big to be dealt with by this algorithm, looking at the self-pairedness of suborbits is not the path we follow.

C Using Magma to show non-existence of a distance-transitive graph

When one wants to see if there exists a distance-transitive graph which has Alt_6 as its automorphism group and in which Sym_4 (which is a maximal subgroup of Alt_6) is the stabilizer of some vertex in this graph, using pencil and paper will (hopefully) be enough to come up with the graph $J(6,2)$ (this is the Johnson graph: the vertices are all 15 unordered pairs from the set $\{1,\ldots,6\}$, two vertices are connected if and only if their intersection is empty (see e.g. [17])) as a distance-transitive graph. However, in more complex cases, where the groups are very large, like in this thesis, a computer program like MAGMA , can be very useful to aid in the computations. A variety of functions and options are at hand to do nice calculations on these groups, which help to obtain more insight into possible graph structures. However, using a computer program to do calculations also brings in a factor of uncertainty about results, since software might be bugged. Since I was the first user (tester) of the recently developed LIE THEORY package in MAGMA , I had to deal with a number of problems, that I will describe in this section.

C.1 Encountered problems

In this section I will discuss the main problems that I encountered using MAGMA (see [1]) for my research. The version of MAGMA involved is V2.10-15.

First there was a number of errors in MAGMA functions that I noticed, resolved and/or notified MAGMA about.

- The function `Centralizer(L,K)` for a Lie algebra L and a subalgebra K of L was very inefficient. Its running time was exponential; I adapted this function to run in polynomial time. Scott H. Murray implemented this new version of the centralizer function into MAGMA and it is now available.
  It is now also possible to compute `Centralizer(L,l)` directly for some Lie algebra L and an element $l \in L$.

- The function `SolvableRadical(M)` would not work correctly in all cases. One of the cases it which it would return an error, was when $M = \mathrm{SolvableRadical}(M)$. This problem is resolved in the new release of MAGMA .

- There existed two functions `SemisimpleLieAlgebra(N,k)` and `SemiSimpleLieAlgebra(N,k)` that would, without further specification return different objects, when called with the same arguments. The former would return the adjoint Lie algebra and the latter would return the simply connected Lie algebra. In the new release of MAGMA only the first function `SemisimpleLieAlgebra(N,k)` exists, using a parameter to determine which Lie algebra will be returned. It standards returns the adjoint Lie algebra. The implementation of the new version is done by Scott. H. Murray.

- The MAGMA manual (see [1]) did not give correct descriptions of some functions (for instance `Weylgroup(L)`, `IsAbelian(L)`) and covered functions that are not recognized by MAGMA (e.g. `ReductiveLieAlgebra(R,k)`, `SetNormalizing( G,BoolElt)`). I notified MAGMA about these errors.

Then there are some functions/options that MAGMA lacks.

- Twisted groups of Lie type have not yet been implemented, though a standard implementation can be very useful. This is work that is currently in progress and will hopefully be available soon. For my calculations, I created a twisted group of type A_7 as a subgroup of a group of type E_7 manually.

Even though computers can perform calculations much faster than humans can, unfortunately there are problems that are still much too big for them. In this thesis I often reached the limits of what the computer could compute/store, forcing me to think of different approaches to tackle my problem. A program like MAGMA will always remain a tool, that has to receive intelligent input, before it is able to return an answer. Still it shows how important it is that functions are optimized for speed, and mathematical structures are optimized for storage space.

C.2 Testing Magma functions for constructing L_{E_7} of adjoint isogeny type over GF(2)

I want to say something about the theoretical concept of the isogeny type of a root datum, which can be used to construct different Lie groups of the same Cartan type. In MAGMA this concept is also used to construct different Lie algebras of the same Cartan type (but corresponding to root data of different isogeny type). See [1]. For the Lie algebra L_{E_7} over $GF(2)$, I checked whether the Lie algebra constructed for a root datum of adjoint isogeny type is really a Lie algebra. Moreover I checked if the corresponding Lie group is really a Lie group and acts on the Lie algebra in the correct way.

There are more ways to create a Lie algebra in MAGMA . One can create a so-called *Simply Connected* version or an *Adjoint* version. See for more theoretical information on this, Section 4.3. It is well known that the Lie algebra L_{SC} which has the Chevalley basis is indeed a Lie algebra over an arbitrary field, hence also over $GF(2)$. Our concern lies with the status of the Lie algebra L_{Ad}. In MAGMA we can set up an Adjoint Lie algebra with its corresponding Lie group and its representation by the following series of commands:

```
R := RootSystem("E7" : Isogeny := "Ad");
L_Ad := LieAlgebra(R, GF(2));
G_Ad := GroupOfLieType(R, GF(2));
ρ := AdjointRepresentation(G);
```

The question we'd like to address here is whether the functions creating L_{Ad}, the corresponding Lie group G_{Ad} and its representation ρ on L_{Ad} are defined correctly when working over the field $GF(2)$. In order to check this we need to check four conditions.

1. L_{Ad} is indeed a Lie algebra;
2. the elements of the Lie group are well defined elements over the field $GF(2)$;
3. the generators of the corresponding Lie group G_{Ad} satisfy the Steinberg relations;
4. the action of the Lie group on the Lie algebra, through the representation ρ leaves the Lie bracket $[\cdot,\cdot]$ invariant.

Ad 1: In order to check that L_{Ad} is indeed a Lie algebra, with basis $\mathcal{B}_{Ad}$, we check whether its basis elements satisfy the relations that are defining for a Lie algebra.

- $[x, x] = 0$ for all $x \in \mathcal{B}_{Ad}$
- $[x, y] + [y, x] = 0$ for all $x, y \in \mathcal{B}_{Ad}$
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathcal{B}_{Ad}$

Checking in MAGMA that these relations hold for the basis elements $\mathtt{L_{Ad}.1}, \dots, \mathtt{L_{Ad}.133}$ is straightforward and shows that L_{Ad} is indeed a Lie algebra.

Ad 2: In general Lie group elements $X_r(1)$ are defined by $X_r(1) := \exp(\mathrm{ad}_{x_r})$ and writing this expression out gives $X_r(1) = 1 + \mathrm{ad}_{x_r} + 1/2(\mathrm{ad}_{x_r})^2 + 1/6(\mathrm{ad}_{x_r})^3 + \dots$. Thus we need to check for each Lie group element $X_i(1) \in G_{Ad}$ that it is well defined. This means that all the terms of $X_i(1)$ should be integers. Let L_{rats} be the Lie group of type E_7 over the rationals. Using

MAGMA we compute the adjoint matrix of all the basis elements of L_{rats} that are not in the Cartan subalgebra and find that their squares are all equal to zero modulo 2, meaning that for all i, $X_i(1)$ only consists of the first two terms which are automatically integers. The fractions that occur in later terms are all cancelled by multiplication with zero.

Ad 3: The group G_{Ad} is generated by the elements $X_1(1), X_2(1), \ldots, X_{126}(1)$. In MAGMA these elements are obtained by $\texttt{X}_\texttt{i}(\texttt{1}) := \texttt{elt} < \texttt{G}_\texttt{Ad}| < \texttt{i}, \texttt{1} >>$ for $i = 1, \ldots, 126$. If these elements satisfy the Steinberg relations G_{Ad} is indeed a Lie group. In [7] we find the following Steinberg relations

- $X_r(t_1)X_r(t_2) = X_r(t_1 + t_2)$
  Over $GF(2)$ this relation is trivially satisfied if $t_1 = 0$ or $t_2 = 0$ since $X_r(0) = 1$. Now assume that $t_1 = t_2 = 1$. The only relation to check is then that $X_i(1)X_i(1) = 1$ which is easily shown to hold for the basis elements using MAGMA .

- $[X_s(u), X_r(t)] = \prod_{i,j>0} X_{ir+js}(C_{ijrs}(-t)^i u^j)$
  If $u = 0$ or $t = 0$, this relation is trivially satisfied. Now assume $u = t = 1$. The relation can be simplified to $[X_s(u), X_r(t)] = \prod_{i,j>0} X_{ir+js}(C_{ijrs})$. Looking at the coefficients C_{ijrs} allows us to simplify this computation even more. According to [7] $C_{32rs} = 1/3M_{r+s,r,2} = 1 \bmod 2$ and $C_{23rs} := 2/3M_{s+r,s,2} = 0 \bmod 2$. Furthermore $C_{i1rs} = M_{rsi} = \binom{p+i}{i}$ where p is the largest integer such that $s - pr$ is a root and $C_{1jrs} = M_{srj} = \binom{q+j}{j}$ where q is the largest integer such that $r - qs$ is a root. These coefficients are easily computed using MAGMA (or even by hand) and these relations indeed hold for the generators of the group G_{Ad}.

- $H_r(t_1)H_r(t_2) = H_r(t_1t_2), \qquad t_1t_2 \neq 0$
  The condition $t_1t_2 \neq 0$ implies we'll only need to check the relations $H_r(1)H_r(1) = H_r(1)$. Since $H_r(t) = n_r(t)n_r(-1)$ and $n_r(t) = X_r(t)X_{-r}(-t^{-1})X_r(t)$ we have $H_r(1) = n_r(1)n_r(1) = X_r(1)X_{-r}(1)X_r(1)X_r(1)X_{-r}(1)X_r(1)$. Since all the basis elements $X_i(1)$ satisfy the first Steinberg relation, we can rewrite $H_r = 1$ by recursively rewriting the middle product on the right hand side of the expression to 1. Thus this relation is trivially satisfied once the first Steinberg relation is satisfied. There are no extra checks needed in MAGMA .

Ad 4: The basis elements $X_i(1)$ act from the right on the basis elements of the Lie algebra through the representation ρ. It needs to be checked though, if this action respects the Lie bracket, thus if

$$[L.i\rho(X_k(1)), L.j\rho(X_k(1))] = [L.i, L.j]\rho(X_k(1))$$

for all $i, j = 1, \ldots 133$ and $k = 1, \ldots, 126$. This again is a very straightforward check. Also this condition is easily shown to hold by computations in MAGMA .

We can conclude that the implementation in MAGMA of the so-called adjoint Lie algebra works fine over GF(2).

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